Partial classification of Moore bipartite graphs

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Abstract

We partially classify Moore bipartite graphs. We prove that a Moore bipartite graph G exists only when the diameter is 2, 3, 4 or 6. However, for the diameters 3, 4 and 6 a full classification is missing; only graphs of degree k, where $k-1$ is a prime power, have been constructed [1, 4]. Similarly to the proofs of the non-existence of Moore graphs for $k \geq 3$ and $D \geq 3$ given in [2, 6] and the non-existence proofs of Moore bipartite graphs for $k \geq 3$ and $D = 5$, $D \geq 7$ presented in [2, 11], our proof relies on the integrality of the multiplicity of an eigenvalue. We prove that, unless $D = 2, 3, 4$ or 6, the multiplicity of some eigenvalue other than $\pm k$ of the adjacency matrix of G is not an integer. Almost nothing is new in our approach, but we want to show the strength of equitable partitions and walk-regularity in tackling these sort of problems, approach followed by Godsil in [6]. Our emphasis is on the clarity of the presentation, and we believe these notes may have some methodological and pedagogical value.

1 Introduction

A general upper bound for the maximum number $N^b_{\Delta,D}$ of vertices in a bipartite graph of maximum degree Δ and diameter D is given by the so-called Moore bipartite bound, denoted by $M^b_{\Delta,D}$.

The fact that $N^b_{\Delta,D}$ is well-defined for any $\Delta \geq 2$ and $D \geq 2$ can be seen by considering the (Δ, D)-*broom graph*, a path of length $D-1 \geq 1$ with $\Delta - 1 \geq 1$ additional vertices connected to one of its ends.

To deduce the Moore bipartite bound, we can use the standard decomposition for a graph of even girth with respect to an edge ab. Let ab be an edge of a bipartite graph G of maximum degree Δ and diameter D. Define the sets A_i and B_i for $0 \leq i \leq D-1$ as follows.

$$\begin{aligned} A_i &= \{c \in V(G) | d(a,c) = i, d(b,c) = i+1\} \\ B_i &= \{c \in V(G) | d(b,c) = i, d(a,c) = i+1\} \end{aligned}$$

The decomposition of G into the sets A_i and B_i is called the *standard decomposition for a graph of even girth with respect to the edge ab* [3].

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Since G is bipartite, its girth $g(G)$ is even, and $A_i \cap B_i = \emptyset$ for $0 \leq i \leq D-1$. Let the edge ab be in a cycle of length $g(G)$. Then, $|A_1|(|B_1|) \leq \Delta - 1$ and $|A_i|(|B_i|) \leq (\Delta-1)|A_{i-1}|(|B_{i-1}|)$ for all i such that $2 \leq i \leq \frac{g(G)}{2} - 1$, and consequently, we have $|A_i|(|B_i|) \leq (\Delta-1)^i$ for $1 \leq i \leq \frac{g(G)}{2} - 1$.

Since $g(G) \leq 2D$, it follows that

$$\begin{aligned}\sum_{i=0}^{D-1} |A_i| + \sum_{i=0}^{D-1} |B_i| &\leq 2\left(1 + (\Delta-1) + (\Delta-1)^2 + \cdots + (\Delta-1)^{D-1}\right) \\ &= \begin{cases} 2\frac{(\Delta-1)^D - 1}{\Delta - 2} & \text{if } \Delta > 2 \\ 2D & \text{if } \Delta = 2 \end{cases} \end{aligned} \tag{1}$$

The right-hand side of Equation (1) is the *Moore bipartite bound.* A bipartite graph of degree Δ, diameter D, and order equal to $M^b_{\Delta,D}$ is called a *Moore bipartite graph.* It can be easily seen that such a graph is regular of degree Δ and girth $2D$.

The Moore bipartite bound represents not only an upper bound on the number of vertices of a bipartite graph of maximum degree Δ and diameter D, but it is also a lower bound on the number $n^e_{\Delta,g}$ of vertices of a regular graph G of degree Δ and girth $g = 2D$ [2].

In the latter context if G has order $M^b_{\Delta,D}$ then G is the incidence graph of a generalized D-gon of order $\Delta - 1$. Incidence graphs of generalized D-gons of order $\Delta - 1$ and Moore bipartite graphs of maximum degree Δ and diameter D are different names for the same class of graphs.

As the graphs we will deal with are regular, we will use k rather than Δ to denote their degrees.

For $k = 2$ and $D \geq 2$ Moore bipartite graphs are the cycles on $2D$ vertices. When $k \geq 3$ the rarity of Moore bipartite graphs was settled by Feit and Higman [5] in 1964, and independently, by Singleton [11] in 1966. They proved that such graphs exist only if the diameter is 2, 3, 4 or 6. Our proof of this assertion relies on the use of equitable partitions [7, 9] and the fact that Moore bipartite graphs are walk-regular graphs, in particular, distance-regular graphs.

For $D = 2$ and each $k \geq 3$ the Moore bipartite graphs of degree k are the complete bipartite graphs of degree k. For $D = 3, 4, 6$ Moore bipartite graphs of degree k have been constructed only when $k - 1$ is a prime power [1].

The question of whether or not Moore bipartite graphs of diameter 3, 4 or 6 exist for other values of k remains open, and represents one of the most famous problems in combinatorics.

2 Notation and Terminology

The vertex set V of a graph G is denoted by $V(G)$, its edge set by $E(G)$, and its diameter $D(G)$.

The set of vertices at distance i from a vertex x in G is denoted by $N_i(x)$. The distance between vertices u and v is denoted by $d(u, v)$.

Given $A, B \subseteq V(G)$, a path $P = x_0 \dots x_l$ is called an $A - B$ *path* if $V(P) \cap A = \{x_0\}$ and $V(P) \cap B = \{x_l\}$. We write $a - b$ path instead of $\{a\} - \{b\}$ path.

For a matrix M, M^T denotes its transpose, and $\Psi_M(x)$ its characteristic polynomial (if M is the adjacency matrix $A(G)$ of a graph G, we may use $\Psi_G(x)$ instead). The identity matrix of order n is denoted by I_n.

The multiplicity of θ as an eigenvalue of $A(G)$ is denoted by $\mathrm{m}(\theta)$.

3 Equitable Partitions

This section is mainly based on [7, Chapter 5] and from [9, Chapter 9].

Let $\pi = \{C_1, \dots, C_r\}$ be a partition of the vertex set of a graph G. We call the subsets C_i *cells*. A partition $\pi = \{C_1, \dots, C_r\}$ is equitable if, for any vertex $u \in C_i$, $|N(u) \cap C_j| = c_{ij}$, viz., $|N(u) \cap C_j|$ is independent of the selection of u. The quotient of G under π, denoted by G/π, is the digraph with the r cells of π as its vertices and c_{ij} arcs from the vertex C_i to the vertex C_j.

For a partition π we define the *characteristic matrix* $P(\pi)$ to be the $|V(G)| \times r$ matrix such that

$$(P)_{ij} = \begin{cases} 1 & \text{if the vertex } i \text{ belongs to the cell } C_j \\ 0 & \text{otherwise} \end{cases}$$

For each $u \in V(G)$ let Π_u be the set of equitable partitions having $C_u = \{u\}$ as their first cell, and Π_1 the set of equitable partitions having a singleton C_1 as their first cell. From now on, let π denote an equitable partition, and let $P := P(\pi)$, $H := G/\pi$, $H_u := G/\pi_u$ if $\pi_u \in \Pi_u$, and $n = |G|$.

Note that P^TP is diagonal, and nonsingular since $(P^TP)_{ii} = |C_{ii}|$, where $|C_{ii}| \geq 1$.

Lemma 3.1 ([9, Lemma 9.3.1]) *Let π be an equitable partition of the graph G, with characteristic matrix P. Then $A(G)P = PA(H)$.*

Lemma 3.2 ([7, Lemma 5.2.2]) *(a) If $A(H)x = \theta x$ then $A(G)Px = \theta Px$.*

(b) If $A(G)y = \theta y$ then $y^TPA(H) = \theta y^TP$.

(c) $\Psi_H(x)$ divides $\Psi_G(x)$.

Lemma 3.3 ([7, Lemma 5.3.1]) *The number of $C_i - C_j$ walks of length l in G is equal to $|C_i|$ times the number of $i - j$ walks of length l in H.*

Corollary 3.1 ([7, Corollary 5.3.2])

$$\frac{(A(H)^l)_{ij}}{(A(H)^l)_{ji}} = \frac{|C_j|}{|C_i|}.$$

Corollary 3.2 ([7, Corollary 5.3.3]) *For $u \in V(G)$, let $\pi_u \in \Pi_u$. Then*

$$\frac{\Psi_{G \setminus \{u\}}(x)}{\Psi_G(x)} = \frac{\Psi_{H_u \setminus \{C_u\}}(x)}{\Psi_{H_u}(x)}.$$

Suppose that for each vertex $u \in V(G)$ there is an equitable partition $\pi_u \in \Pi_u$. If $x^T = (x_1, \ldots, x_n)$ is an eigenvector of $A(G)$ with respect to the eigenvalue θ then $x^T P$ will be a left eigenvector of $A(H_u)$ with respect to the eigenvalue θ iff $x^T P \neq 0$. If $x^T P = 0$ then $x_u = 0$. As not all components of x can be zero, there exists at least one $v \in V(G)$ such that $x^T P \neq 0$, and thus, θ is an eigenvalue of $A(H_v)$.

Theorem 3.1 ([7, Theorem 5.3.4]) *Suppose that for each vertex $u \in V(G)$ there is an equitable partition $\pi_u \in \Pi_u$. If θ is an eigenvalue of G, then*

$$\mathrm{m}(\theta) = \lim_{x \to \theta} \frac{\Psi'_G(x)(x-\theta)}{\Psi_G(x)} = \lim_{x \to \theta} \sum_{u \in V(G)} \frac{\Psi_{H_u \setminus \{C_u\}}(x)(x-\theta)}{\Psi_{H_u}(x)}.$$

Corollary 3.3 ([8, Corollary 3.6]) *For each eigenvalue θ of G, there is at least one $u \in V(G)$ such that for any $\pi_u \in \Pi_u$ we have that θ is an eigenvalue of $A(H_u)$.*

3.1 Walk-regular Graphs

We say that a graph G is *walk-regular* if, for any vertices $u, v \in V(G)$, $\Psi_{G/\{u\}}(x) = \Psi_{G/\{v\}}(x)$.

Let $W_{uv}(G, x) = \sum_{m \geq 0} (A^m(G))_{uv} x^m$, that is, $W_{uv}(G, x)$ denotes the walk generating function counting the walks starting at the vertex u and finishing on the vertex v.

Lemma 3.4 ([7, It follows from Lemma 4.1.1]) *Let $u \in V(G)$. Then,*

$$W_{uu}(G, x) = x^{-1} \frac{\Psi_{G/\{u\}}(x^{-1})}{\Psi_G(x^{-1})}$$

From Lemma 3.4 it follows that $x^{-1} W_{uu}(G, x^{-1}) = \frac{\Psi_{G/\{u\}}(x)}{\Psi_G(x)}$, so for a walk-regular graph G, $W_{uu}(G, x^{-1})$ is independent of the selection of u.

Therefore, G is walk-regular if, for any $u \in V(G)$, the number of closed walks starting at u is independent of u, that is, $A^m(G)$ has a constant diagonal for any $m \in \mathbb{N}$.

Using Corollary 3.2, if G is a walk-regular graph, Theorem 3.1 can be simplified as follows.

Theorem 3.2 ([7, Corollary 5.3.4]) *Let G be a walk-regular graph, π any partition in Π_1, and θ an eigenvalue of G. Then*

$$\frac{\Psi'_G(x)}{\Psi_G(x)} = n \frac{\Psi_{H \setminus \{C_1\}}(x)}{\Psi_H(x)}$$

and therefore,

$$\mathrm{m}(\theta) = \lim_{x \to \theta} n \frac{\Psi_{H \setminus \{C_1\}}(x)(x-\theta)}{\Psi_H(x)}$$

or equivalently,

$$\mathrm{m}(\theta) = n \frac{\Psi_{H \setminus \{C_1\}}(\theta)}{\Psi'_H(\theta)}$$

(setting $\Psi_H(x) = (x-\theta) f(x)$, it follows that $\Psi'_H(x) = (x-\theta) f'(x) + f(x)$, and thus, $\Psi'_H(\theta) = f(\theta)$).

3.1.1 Distance-regular Graphs

An important class of walk-regular graphs is the class of distance-regular graphs. A graph is *distance-regular* if, for any $u, v \in V(G)$ and any $i \in \mathbb{N}$, the number $|N(u) \cap N_i(v)|$ depends only on $d_G(u, v)$. We define the *distance partition* of a graph G to be $\{N_0(u), \ldots, N_{D(G)}(u)\}$. Alternatively, a graph G is distance-regular if

(a) for each $u \in V(G)$, the distance partition is equitable, and

(b) the isomorphism from H_u to H_v maps u to v.

Let G be a distance regular, π_u the distance partition with respect to u, $a_i = |N(u) \cap N_i(v)|$, $b_i = |N(u) \cap N_{i+1}(v)|$, and $c_i = |N(u) \cap N_{i-1}(v)|$, when $d(u, v) = i$. Then

$$A(H_u) = \begin{pmatrix} 0 & b_0 & & & & & & \\ c_1 & a_1 & b_1 & & & & 0 & \\ & c_2 & a_2 & b_2 & & & & \\ & & c_3 & a_3 & b_3 & & & \\ & & & & & \ddots & & \\ & & & & \ddots & \ddots & & \\ & 0 & & & & c_{D-1} & a_{D-1} & b_{D-1} \\ & & & & & & c_D & a_D \end{pmatrix}$$

Note that for any two vertices $u, v \in V(G)$, we have that $A(H_u) = A(H_v) = A(H)$. Therefore, by Corollary 3.3 and Lemma 3.2(c), we obtain the following.

Proposition 3.1 *The minimal polynomials of $A(H)$ and $A(G)$ coincide.*

4 Moore Bipartite Graphs

From now on, let G denote a Moore bipartite graph of degree k and diameter D.

Proposition 4.1 ([2, Proposition 23.1(2)]) *Let $\pi = \{C_1, \ldots, C_D\}$ be a distance partition of $V(G)$. Then, π is equitable and G is distance-regular with the following quotient matrix:*

$$A(H) = \begin{pmatrix} 0 & k & & & & & & \\ 1 & 0 & k-1 & & & & 0 & \\ & 1 & 0 & k-1 & & & & \\ & & 1 & 0 & k-1 & & & \\ & & & & & \ddots & & \\ & & & & \ddots & \ddots & & \\ & 0 & & & & 1 & 0 & k-1 \\ & & & & & & k & 0 \end{pmatrix}$$

Proof. It suffices to prove that G is distance regular. Let $u, v \in V(G)$ such that $d(u,v) = i$ for $1 \le i \le D$. Clearly, $a_i = 0$ for $1 \le i \le D$. For $1 \le i \le D-1$ $c_i = 1$, otherwise in G there will a cycle of length at most $2D-2$. For $i = D$, as G is bipartite, $c_D = k$. For $1 \le i \le D-1$, as G is bipartite of girth $2D$, $b_i = k-1$ and $b_D = 0$. □

By Proposition 3.1, any eigenvalue θ of $A(H)$ is an eigenvalue of $A(G)$. Our aim is now to prove that $\mathrm{m}(\theta)$ is not an integer for some θ. We first need to find a formula for $\mathrm{m}(\theta)$.

For this purpose, we define polynomials $p_m(x)$ as follows: $p_0(x) = k$ and $p_1(x) = x$.

$$p_{m+2}(x) = xp_{m+1}(x) - (k-1)p_m(x) \text{ for } m \ge 0 \tag{2}$$

The polynomial $p_m(x)$ for $1 \le m \le D$ equals the characteristic polynomial of the matrix formed by the entries in the last m rows and columns of $A(H)$; this fact can be seen by applying Laplace expansion of determinants. Then $p_D(x) = \Psi_{H\setminus\{C_1\}}(x)$.

Furthermore, by using Laplace expansion on row 1 of $A(H)$, we obtain that

$$\Psi_H(x) = xp_D(x) - kp_{D-1}(x) \tag{3}$$

Now we want to find an analytic expression for the polynomial $p_m(x)$. Here, the simplest way to go is to write the following in *Mathematica* [13], although $p_m(x)$ will not be given in a simplified way.
"FullSimplify[RSolve[p[m + 2] == x*p[m + 1] - (k - 1)*p[m], p[0] == k, p[1] == x, p[m], m]]."
However, we proceed by using ordinary power series [12], which will prove very beneficial in the end if we consider the simplified expression we will obtain for $p_m(x)$. Let $P(x,t) = \sum_{m\ge 0} p_m(x)t^m$.

$$\begin{aligned}
P(x,t) &= k + xt + \sum_{m\ge 0} p_{m+2}(x)t^{m+2} \\
&= k + xt + \sum_{m\ge 0} (xp_{m+1}(x) - (k-1)p_m(x))t^{m+2} \text{ (Using Equation 2)} \\
&= k + xt + xt\sum_{m\ge 0} p_{m+1}(x)t^{m+1} - (k-1)t^2\sum_{m\ge 0} p_m(x)t^m \\
&= k + xt + xt(P(x,t) - k) - (k-1)t^2P(x,t) \\
&= \frac{k-(k-1)xt}{1 - xt + (k-1)t^2}
\end{aligned}$$

Having some known power series at hand [6], we have that

$$\sum_{m\ge 0} \frac{\sin(m+1)\alpha}{\sin\alpha} s^m = \frac{1}{1 - 2\cos\alpha s + s^2} \quad \text{where } \sin\alpha \ne 0$$

So substituting $q = \sqrt{k-1}$ and $t = \frac{s}{q}$ into $P(x,t)$, we obtain that $P(x,s) = \frac{k-qxs}{1-\frac{xs}{q}+s^2}$.

As our aim is to compute multiplicities of eigenvalues of $A(H)$, we can assume the variable x represents an eigenvalue of $A(H)$. We now suppose that some eigenvalue θ of $A(H)$ satisfy $|\theta| < 2q$. We will prove that the multiplicity of such eigenvalues cannot be an integer. Note that, as G is bipartite, $\pm k$ are eigenvalues that do not satisfy the inequality. So, setting $x = 2q\cos\alpha$, for $0 < \alpha < \pi$ (since $\sin\alpha \neq 0$), it follows that

$$P(2q\cos\alpha, s) = \frac{k}{1 - 2\cos\alpha s + s^2} - \frac{2q^2\cos\alpha s}{1 - 2\cos\alpha s + s^2}$$

and thus,

$$\frac{p_m(2q\cos\alpha)}{q^m} = \frac{k\sin(m+1)\alpha - 2q^2\cos\alpha\sin m\alpha}{\sin\alpha} \tag{4}$$

Using Equations (3) and (4), we obtain that

$$\begin{aligned}\Psi_H(2q\cos\alpha) &= (2q\cos\alpha)q^D\frac{k\sin(D+1)\alpha - 2q^2\cos\alpha\sin D\alpha}{\sin\alpha} - kq^{D-1}\frac{k\sin D\alpha - 2q^2\cos\alpha\sin(D-1)\alpha}{\sin\alpha}\\ &= \frac{q^{D-1}}{\sin\alpha}\left[2kq^2\cos\alpha\sin(D+1)\alpha - (2q\cos\alpha)^2q^2\sin D\alpha - k^2\sin D\alpha + 2kq^2\cos\alpha\sin(D-1)\alpha\right]\end{aligned}$$

Setting $g(2q\cos\alpha) := 2kq^2\cos\alpha\sin(D+1)\alpha - (2q\cos\alpha)^2q^2\sin D\alpha - k^2\sin D\alpha + 2kq^2\cos\alpha\sin(D-1)\alpha$, we have

$$\Psi_H(2q\cos\alpha) = \frac{q^{D-1}}{\sin\alpha}g(2q\cos\alpha)$$

Lemma 4.1 *$A(H)$ has $D+1$ distinct eigenvalues:*

$$\pm k, \quad \theta_i = 2q\cos\frac{i\pi}{D} \;\; \textit{for } i \in \{1, 2, \ldots, D-1\}$$

where $q = \sqrt{k-1}$.

Proof. We first prove that $\theta \neq \pm k$ is an any eigenvalue of $A(H)$ iff $\sin D\alpha = 0$.

$$\begin{aligned}g(2q\cos\alpha) &= 2kq^2\cos\alpha\sin(D+1)\alpha - (2q\cos\alpha)^2q^2\sin D\alpha - k^2\sin D\alpha + 2kq^2\cos\alpha\sin(D-1)\alpha\\ &= 2kq^2\cos\alpha\sin(D+1)\alpha + (2q\cos\alpha)^2\sin D\alpha - k(2q\cos\alpha)^2\sin D\alpha - k^2\sin D\alpha +\\ &\quad + 2kq^2\cos\alpha\sin(D-1)\alpha\\ &= 2kq\cos\alpha\left[q\left(\sin(D+1) + \sin(D-1)\alpha\right) - 2q\cos\alpha\sin D\alpha\right] + ((2q\cos\alpha)^2 - k^2)\sin D\alpha\\ &= 2kq\cos\alpha\left[q\left(2\cos\alpha\sin D\alpha\right) - 2q\cos\alpha\sin D\alpha\right] + ((2q\cos\alpha)^2 - k^2)\sin D\alpha\\ &= ((2q\cos\alpha)^2 - k^2)\sin D\alpha\end{aligned}$$

As $|\theta| < 2q$, it follows that $(2q\cos\alpha)^2 - k^2 \neq 0$, and $\sin D\alpha = 0$. Solving the equation $\sin D\alpha = 0$ for $0 < \alpha < \pi$, we get the following $D-1$ distinct solutions $\alpha_i = \frac{i\pi}{D}$, where $i \in \{1, \ldots, D-1\}$. □

Thus,

$$\Psi_H(2q\cos\alpha) = \frac{q^{D-1}}{\sin\alpha}((2q\cos\alpha)^2 - k^2)\sin D\alpha \tag{5}$$

Theorem 4.1 *The multiplicity of $\theta = 2q\cos\alpha$ as an eigenvalue of $A(H)$ is*

$$\mathrm{m}(\theta) = -n\frac{k(\theta^2 - 4q^2)}{2D(\theta^2 - k^2)}.$$

Proof. By Theorem 3.2, we have $\mathrm{m}(\theta_i) = n\frac{p_D(\theta)}{\Psi'_H(\theta)}$, so we need to compute $\Psi'_H(\theta)$.

$$\begin{aligned}\frac{\mathrm{d}\Psi'_H(\theta)}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{d}\alpha} &= \frac{\mathrm{d}}{\mathrm{d}\alpha}\left(\frac{q^{D-1}}{\sin\alpha}((2q\cos\alpha)^2 - k^2)\sin D\alpha\right) \\ &= q^{-1+D}\left(\frac{k^2}{\sin\alpha}(-D\cos D\alpha + \cot\alpha\sin D\alpha) + 4q^2\cos\alpha\left(D\cos D\alpha\cot\alpha - (2+\cot^2\alpha)\sin D\alpha\right)\right)\end{aligned}$$

So $\Psi'_H(\theta) = \frac{q^{-1+D}}{-2q\sin\alpha}\left(\frac{k^2}{\sin\alpha}(-D\cos D\alpha + \cot\alpha\sin D\alpha) + 4q^2\cos\alpha\left(D\cos D\alpha\cot\alpha - (2+\cot^2\alpha)\sin D\alpha\right)\right)$. Using the fact that $\sin D\alpha = 0$, we obtain that $\Psi'_H(\theta) = -\frac{q^{-2+D}D\cos D\alpha}{2\sin^2\alpha}\left(-k^2 + 4q^2\cos^2\alpha\right)$. As $\sin D\alpha = 0$, $p_D(\theta) = q^D\frac{k\sin(D+1)\alpha}{\sin\alpha}$, and if we use $\sin(D+1)\alpha = \sin D\alpha\cos\alpha + \sin\alpha\cos D\alpha$, we finally get that $p_D(\theta) = kq^D\cos D\alpha$.

$$\mathrm{m}(\theta) = \frac{p_D(\theta)}{\Psi'_H(\theta)} = -n\frac{2q^2k\sin^2\alpha}{D(4q^2\cos^2\alpha - k^2)} = -n\frac{k(4q^2\cos^2\alpha - 4q^2)}{2D(4q^2\cos^2\alpha - k^2)} = -n\frac{k(\theta^2 - 4q^2)}{2D(\theta^2 - k^2)}.$$

□

Before stating our main theorem, we present a very known fact on the rationality of the cosine function.

Theorem 4.2 ([10, Corollary 3.12]) *If α is rational in degrees, say $\alpha = 2\pi r$ for some rational number r, then the only rational values of $\cos\alpha$ are 0, $\pm 1/2$ and ± 1.*

Theorem 4.3 *If there is a Moore bipartite graph of diameter D then $D \in \{2, 3, 4, 6\}$.*

Proof. We proceed by showing that $\mathrm{m}(\theta)$ is not an integer unless $D \in \{2, 3, 4, 6\}$. By the expression of $\mathrm{m}(\theta)$ we see that $\mathrm{m}(\theta) \in \mathbb{N} \Rightarrow \theta \in \mathbb{Q} \Leftrightarrow \cos^2\alpha \in \mathbb{Q}$. Let θ_1 be as in Lemma 4.1, then $\theta_1 = 2q\cos\frac{\pi}{D}$. Using $\cos 2\frac{\pi}{D} = 2\cos^2\frac{\pi}{D} - 1$, it follows that $\cos 2\frac{\pi}{D} \in \mathbb{Q}$, and by Theorem 4.2, it follows that $D \in \{2, 3, 4, 6\}$, as required. □

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