Classification of graphs of maximum degree Δ, diameter 2, and cyclic defect

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Abstract

We classify the graphs of maximum degree Δ, diameter 2, and cyclic defect. The exposition is mainly based on Fajtlowicz's original proof [5].

1 Introduction

It is known that the Moore bound, denoted by $M_{\Delta,D}$ and defined below, represents an upper bound on the order of a graph of maximum degree Δ and diameter D [2].

$$M_{\Delta,D} = \begin{cases} 1+\Delta\,\frac{(\Delta-1)^D-1}{\Delta-2} & \text{if } \Delta>2 \\ 2D+1 & \text{if } \Delta=2 \end{cases} \tag{1}$$

Moore graphs (graphs whose order equals the Moore bound) exist only for $D=2$, in which case $\Delta=2,3,7$ and possibly 57 [6, 1]. It was asked in [4]: Given non-negative numbers Δ and δ (*defect*), is there a graph of maximum degree Δ, diameter 2 and order $M_{\Delta,2}-\delta$, that is, a $(\Delta,2,-\delta)$-graph? The case $\delta=1$ was solved by Erdös *et al.* [4]; with the exception of C_4, there is no $(\Delta,2,-1)$-graph. Here we consider the defect 2.

It is not difficult to see that if $\delta<\Delta$ then a $(\Delta,2,-\delta)$-graph must be regular. Let Γ be a simple graph of maximum degree Δ, diameter 2 and order $n=M_{\Delta,2}-2$. Then, for $\Delta\geq 3$ Γ is regular. For $\Delta\leq 2$ the path on 3 vertices is the only such graph. In the following, assume $\Delta\geq 3$. The

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girth of Γ is 4. Every vertex of Γ is contained in either Θ_2 (the union of three independent paths of length 2 with common endvertices) or in exactly two 4-cycles. If there are at least 2 paths of length at most 2 from a vertex v to a vertex u, then we say that v is a *repeat* of u (or viceversa), and we may denote v by $rep(u)$. Then, in this case, we have two repeats (not necessarily different) for each vertex of Γ. Next we define the *repeat (multi)graph* R_Γ of Γ. R_Γ is the graph with $V(R_\Gamma) = V(\Gamma)$, with two vertices being adjacent iff one is a repeat of the other. Then R_Γ is a union of vertex-disjoint cycles of length at least 2.

Let A be the adjacency matrix of Γ, and let B be the adjacency matrix of R_Γ, called *the defect matrix*, in which the main diagonals consist entirely of 0's, and the row and column sums are equal to 2. With a suitable labeling of Γ, B becomes a direct sum of symmetric a^{th}-order circulants of the form,

$$D_a = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 1 \\ 1 & 0 & 1 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & 0 & 0 & \dots & 1 & 0 \end{pmatrix} a \geq 2$$

This matrix was first studied in [3], in the context of regular graphs of girth 5.

Then the following equation holds.

$$A^2 + A - (\Delta - 1)I_n = J_n + B \tag{2}$$

where J_n is the square matrix of order n whose entries are all 1, and I_n is the identity matrix of order n.

Henceforth we consider the case of $B = C_n$. It is well known that the eigenvalues and their corresponding multiplicities of a matrix representing a n-cycle are

$$\begin{pmatrix} 2 & 2\cos\frac{2\pi}{n} \times 1 & 2\cos\frac{2\pi}{n} \times 2 & \dots & 2\cos\frac{2\pi}{n} \times (\frac{n}{2} - 1) & -2 \\ 1 & 2 & 2 & \dots & 2 & 1 \end{pmatrix} (n \text{ even})$$

$$\begin{pmatrix} 2 & 2\cos\frac{2\pi}{n} \times 1 & 2\cos\frac{2\pi}{n} \times 2 & \dots & 2\cos\frac{2\pi}{n} \times (\frac{n-1}{2}) \\ 1 & 2 & 2 & \dots & 2 \end{pmatrix} (n \text{ odd})$$

The first row displays the eigenvalues, and the second row their corresponding multiplicities.

It is also known that the spectrum of J_n is $\begin{pmatrix} n & 0 \\ 1 & n-1 \end{pmatrix}$.

2 The Möbious ladder on 8 vertices as the only solution for Γ

In this section we prove the following.

Theorem 2.1 ([5]) *If $B = C_n$, then $\Delta = 3$ and Γ is the Möbious ladder on 8 vertices.*

Proof. If a vertex of Γ is contained in a Θ_2, then in B we would have a 2-cycle. Therefore, every vertex of Γ is contained in exactly two 4-cycles. Since the number $\frac{2n}{4}$ of 4-cycles in Γ must be integer, $n \equiv 0 \pmod 2$, and in fact, $n \equiv 0 \pmod 4$.

As A, B and J_n are symmetric matrices, they are diagonalizable. Since J_n commutes with A and B, B commutes with A, and hence, all the three matrices are simultaneously diagonalizable, that is, there is an orthogonal matrix P for which $P^{-1}AP$, $P^{-1}BP$ and $P^{-1}J_nP$ are diagonal, and the columns of P are corresponding eigenvectors for each of these matrices.

We have that the eigenvalue n of J_n is paired with the eigenvalue 2 of B, and Δ of A (all associated to the all 1's vector).

As -2 is a simple eigenvalue of C_n, there is a simple eigenvalue γ of A satisfying

$$\gamma^2 + \gamma - (\Delta - 1) = -2. \tag{3}$$

Since in C_n the eigenspace of -2 contains the vector $u = (1, -1, 1, -1, \ldots)^T$, in A an eigenvector associated with γ has the form αu for $\alpha \in \mathbb{R}$, and consequently, u is also an eigenvector of A, implying that γ must be integer.

As $4|n$, 0 is an eigenvalue of C_n with multiplicity 2. Therefore

$$x^2 + x - (\Delta - 1) = 0. \tag{4}$$

Denote by θ_1 and θ_2 the roots of Equation (4). Suppose θ_1 and θ_2 are rational. The discriminant of Equation (4) is $4\Delta - 3$, and like the one of (3) $4\Delta - 11$, must be a perfect square. The only pair of perfect squares differing by 8 is {1,9}, implying $\Delta = 3$, in which case, Γ is the Möbious ladder on 8 vertices.

It follows that θ_1 and θ_2 are simple and irrational eigenvalues of Γ (they are algebraic conjugates). We complete the proof of Theorem 2.1 by showing that θ_1 and θ_2 cannot be irrational.

In C_n the eigenspace associated with 0 has dimension 2, and the vectors $u = (0, -1, 0, 1, \ldots)^T$ and $v = (-1, 0, 1, 0, \ldots)^T$ form a basis of it. Therefore, in A the vectors in the one-dimensional eigenspaces of θ_1 and θ_2 have the form $\alpha u + \beta v$. Therefore, we can say that the vectors $t_1 = (1, q, -1, -q, \ldots)^T$ and $t_2 = (1, p, -1, -p, \ldots)^T$ are associated with θ_1 and θ_2, respectively. Since eigenvectors with respect to different eigenvalues are orthogonal, we have

$$pq = -1 \tag{5}$$

Since $At_1 = \theta_1 t_1$ and $At_2 = \theta_2 t_2$, by considering the first two rows of A, we find $a, b, c, d \in \mathbb{Z}$ such that

$$a + bq = \theta_1 \tag{6}$$

$$c + dq = \theta_1 q \tag{7}$$

$$a + bp = \theta_2 \tag{8}$$

$$c + dp = \theta_2 p \tag{9}$$

Solving the System of Equations (5), (6), (7), (8) and (9), we have

$$b = c \tag{10}$$

$$a + d = \theta_1 + \theta_2 = -1 \tag{11}$$

In the labeling associated to Γ, denote by z_i the vertex corresponding to the i row of A, and by $w_1^{(i)}$, $w_2^{(i)}$, $w_3^{(i)}$ and $w_4^{(i)}$, the numbers of neighbors of z_i such that in At_1 the corresponding components of t_1 are 1, -1, q and $-q$, respectively. Then, $w_1^{(i)} + w_2^{(i)} + w_3^{(i)} + w_4^{(i)} = \Delta$.

Consider z_1, then $\theta_1 = (w_1^{(1)} - w_2^{(1)}) + (w_3^{(1)} - w_4^{(1)})q = a + bq$. Note that a is even iff $w_1^{(1)} \equiv w_2^{(1)} \pmod 2$, and that b is even iff $w_3^{(1)} \equiv w_4^{(1)} \pmod 2$. Since Δ is odd, $a \equiv b + 1 \pmod 2$, and from Equations (10) and (11), we have

$$c \equiv d \pmod 2 \tag{12}$$

Analogously, considering z_2, we have that $\theta_1 q = (w_1^{(2)} - w_2^{(2)}) + (w_3^{(2)} - w_4^{(2)})q = c + dq$, and that $c \equiv d + 1 \pmod 2$, which contradicts Equation (12). Therefore, θ_1 and θ_2 cannot be irrational, and thus, the theorem follows. $\square$

3 Concluding remarks

Consider a regular graph Γ of degree Δ, girth 5, and order $M_{\Delta,2} + 2$. Define the graph E_Γ on the same vertex set as Γ, with two vertices being adjacent iff their distance is 3. Then, by using the same method exposed here, we can prove the following.

Theorem 3.1 *There is no regular graph of odd degree $\Delta \geq 3$, girth 5, and order $M_{\Delta,2} + 2$, such that $E_\Gamma = C_n$.*

References

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