New benchmarks for large scale networks with given maximum degree and diameter

Eyal Loz*

Department of Mathematics

University of Auckland

Auckland, New Zealand

Guillermo Pineda-Villavicencio†

Centre for Informatics and Applied Optimization

University of Ballarat

Ballarat, Victoria 3353, Australia

Abstract

Large scale networks have become ubiquitous elements of our society. Modern social networks, supported by communication and travel technology, have grown in size and complexity to unprecedented scales. Computer networks, such as the Internet, have a fundamental impact on commerce, politics and culture. The study of networks is also central in biology, chemistry and other natural sciences. Unifying aspects of these networks are a small maximum degree and a small diameter, which are also shared by many network models, such as small-world networks. Graph theoretical methodologies can be instrumental in the challenging task of predicting, constructing and studying the properties of large scale networks. This task is now necessitated by the vulnerability of large networks to phenomena such as cross-continental spread of disease and *botnets* (networks of malware). In this paper we produce the new largest known networks of maximum degree $17 \leq \Delta \leq 20$ and diameter $2 \leq D \leq 10$, using a wide range of techniques and concepts, such as graph compounding, vertex duplication, Kronecker product, polarity graphs and voltage graphs. In this way, we provide

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[*] eyal@eyal.com.au

[†] work@guillermo.com.au

new benchmarks for networks with given maximum degree and diameter, and a complete overview of state-of-the-art methodology that can be used to construct such networks.

Keywords: Degree/diameter problem, Moore bound, Moore graphs, large scale networks, vertex duplication, graph compounding, voltage assignment, polarity graphs, voltage graphs.

1 Introduction

Networks can be regarded as discrete objects, where nodes represent the members of these networks and edges represent connections among these members, and thus are usually modeled and studied using graph theoretical methodologies. Constructing large graphs with bounded *degree* Δ (number of connections attached to a node) and small *diameter* D (length of a shortest path linking any two farthest nodes) is central in the study of large scale networks (LSNs).

The Moore bound $M_{\Delta,D}$ represents an upper bound on the maximum number $N_{\Delta,D}$ of nodes in such a network of degree at most Δ and diameter D; see [7, 40].

$$\begin{aligned} M_{\Delta,D} &= 1+\Delta+\Delta(\Delta-1)+\cdots+\Delta(\Delta-1)^{D-1} \\ &= 1+\Delta(1+(\Delta-1)+\cdots+(\Delta-1)^{D-1}) \\ &= \begin{cases} 1+\Delta\,\frac{(\Delta-1)^D-1}{\Delta-2} & \text{if } \Delta>2 \\ 2D+1 & \text{if } \Delta=2 \end{cases} \end{aligned} \tag{1}$$

Values of $N_{\Delta,D}$ play an important role in predictions of the *order* (number of nodes) that LSNs of maximum degree Δ and diameter D can have. Graphs of maximum degree Δ, diameter D, and order $N_{\Delta,D}$ provide topologies for optimal networks, where “optimality” is interpreted as the largest possible number of nodes in such a LSN.

When predicting the order of a LSN we must also consider the clustering involved in the network. In general, the largest known networks do not have a high level of clustering. In networks of small diameters, however, clustering could be observed. Some real LSNs, however, exhibit a high level of clustering [1, 47]. The order of such networks will therefore be much smaller than the order of the corresponding largest known graphs (most of which have not been shown optimal), implying that these networks are far from being optimal.

The largest known topologies of given maximum degree $3 \leq \Delta \leq 16$ and diameter $2 \leq D \leq 10$ can be found in the online tables [13, 36]. The gap between the current best constructions and the current best theoretical upper bounds is huge. The problem of reducing this gap is very challenging, even for small values of Δ and D. This problem is known as *the degree/diameter problem*, and in graph-theoretical terms it can be stated as follows:

> Given natural numbers Δ and D, find the largest possible number of vertices $N_{\Delta,D}$ in a graph of maximum degree Δ and diameter D.

A graph of maximum degree Δ, diameter D and order $M_{\Delta,D}$ is called a *Moore graph*. Moore graphs exist only for $D = 1, 2$. For diameter $D = 1$ Moore graphs are the complete graphs of order $\Delta + 1$, while for diameter $D = 2$, Moore graphs exist for $\Delta = 2, 3, 7$ and possibly 57, but not for other degrees [2, 15, 33].

Graphs of maximum degree Δ, diameter D and order $M_{\Delta,D} - \epsilon$ for $\epsilon > 0$ have been considered in the literature. The parameter ϵ is called the *defect*. Graphs of defect 1 were completely classified by Bannai and Ito [3], and independently by Kurosawa and Tsujii [34]. The problem of classifying graphs of defect $\epsilon \geq 2$ is largely unexplored. For more information on the degree/diameter problem, the interested reader is referred to [36, 40].

In this paper we give a complete overview of state-of-the-art methodology that can be used in constructing large graph of bounded degree and small diameter. We extend the table of largest known graphs [13, 36], and produce all the new largest known graphs of maximum degree $17 \leq \Delta \leq 20$ and diameter $2 \leq D \leq 10$, using a wide range of techniques and concepts, such as graph compounding, vertex duplication, Kronecker product, polarity graphs and voltage graphs. In this way, we provide new benchmarks for LSNs with given maximum degree $17 \leq \Delta \leq 20$ and diameter $2 \leq D \leq 10$, and sufficient information to allow the implementation of the graphs we provide.

The rest of the paper is structured as follows. Section 2 establishes the notation and the necessary theoretical background to our constructions. A complete description of the new graphs is presented in Section 3, allowing our graphs to be reproduced to model LSNs. Finally, Section 4 summarizes the results obtained in the paper.

2 Terminology and Techniques

The terminology and notation used in this paper is standard and consistent with that used in [20].

The vertex set of a graph Γ is denoted by $V(\Gamma)$, and its edge set by $E(\Gamma)$. For an edge $e = \{u, v\}$, we write uv, or alternatively, $u \sim v$.

The number of vertices of Γ, denoted by $|\Gamma|$, is the order of Γ. In general, $|X|$ denotes the cardinality of a set X.

A *digraph* Λ is a pair (V, A) of sets satisfying $A \subseteq V \times V$, where $V \neq \emptyset$, and $\times$ denotes the cartesian product between sets. The elements of V and A are called the *vertices* and *arcs* of the digraph Λ, respectively.

As before, the number of vertices of Λ, denoted by $|\Lambda|$, is the order of Λ.

For an arc $a = (x, y)$, the first vertex x, denoted by $tail(a)$, is its *tail*; and the second vertex y, denoted by $head(a)$, is its *head*. The head and tail of an arc are its *endvertices* or *ends*.

Let x be a vertex of Λ. The *out-degree* $d^+(x)$ is the number of arcs in Λ with tail x, while the *in-degree* $d^-(x)$ is the number of arcs in Λ with head x.

We call a graph of maximum degree Δ and diameter D a (Δ, D)-*graph*.

2.1 Generalized Polygons

Most of the material presented in this section is from [11].

An *incidence structure* is a triple $\Omega = (\mathcal{P}, \mathcal{L}, \mathcal{I})$, where $\mathcal{P} \neq \emptyset$ is a set of *points*, $\mathcal{L} \neq \emptyset$ is a set of *lines*, $\mathcal{P} \cap \mathcal{L} \neq \emptyset$, and $\mathcal{I} \subseteq \mathcal{P} \times \mathcal{L}$ is a relation, called the *incidence relation*. Given a point p and a line l, if $(p, l) \in \mathcal{I}$ then we say that p and l are *incident*.

Let $\Omega = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ be an incidence structure. The *incidence graph* Γ of Ω is the graph with vertex set $V(\Gamma) = \mathcal{P} \cup \mathcal{L}$, and the following adjacency relation: $\{x, y\} \in E(\Gamma) \leftrightarrow x\mathcal{I}y$ or $y\mathcal{I}x$ for $x, y \in V(\Gamma)$.

A *generalized D-gon* is an incidence structure whose incidence graph is a bipartite graph of diameter D and girth $2D$. It is common to use standard names for small polygons, for instance, generalized quadrangle instead of generalized 4-gon. We denote the incidence graph of a projective plane of order $\Delta - 1$ by $I_{\Delta-1}$, the incidence graph of the symplectic generalized quadrangle of order $\Delta - 1$ by $Q_{\Delta-1}$, and the incidence graph of the classical generalized hexagon of order $\Delta - 1$ by $H_{\Delta-1}$.

A generalized polygon of order (s, t) is called *thick* if $s > 1$ and $t > 1$.

Theorem 2.1 ([24, 30]) *If a generalized D-gon of order (s, t) is thick then there are only the following possibilities.*

(*i*) $D = 2$.

(*ii*) $D = 3$ *and* $s = t$.

(*iii*) $D = 4$, $t \leq s^2$ *and* $s \leq t^2$.

(*iv*) $D = 6$, st *is a perfect square,* $t \leq s^3$ *and* $s \leq t^3$.

(*v*) $D = 8$, $2st$ *is a perfect square,* $t \leq s^2$ *and* $s \leq t^2$.

Theorem 2.2 ([11, Chapter 9]) *Let* $\Omega = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ *be a generalized D-gon of order* (s, t).

(*i*) *If* $D = 3$ *then* $|\mathcal{P}| = |\mathcal{L}| = 1 + s + s^2$.

(*ii*) *If* $D = 2m$ *then* $|\mathcal{P}| = (1 + s)(1 + st + (st)^2 + \ldots + (st)^{m-1})$ *and*
$|\mathcal{L}| = (1 + t)(1 + st + (st)^2 + \ldots + (st)^{m-1})$.

A generalized triangle of order s, $s > 1$, is a *projective plane* of order s.

A projective plane is an incidence structure satisfying the following axioms:

(*i*) Any two distinct points are incident with exactly one common line.

(*ii*) Any two distinct lines are incident with exactly one common point.

(*iii*) There are three pairwise non-collinear points.

A family of thick projective planes of order s is presented in Appendix A.

A *generalized quadrangle of order* (s, t) is an incidence structure satisfying the following axioms:

(*i*) Any two distinct points are incident with at most one common line.

(*ii*) Any two distinct lines are incident with at most one common point.

(*iii*) Any line is incident with exactly $s + 1$ points and any point is incident with exactly $t + 1$ lines;

(*iv*) For any line l and any point p with $(p, l) \notin \mathcal{I}$, there exists a unique pair $(p_1, l_1) \in \mathcal{P} \times \mathcal{L}$ such that $p\mathcal{I}l_1\mathcal{I}p_1\mathcal{I}l$.

The symplectic generalized quadrangle of order s is presented in Appendix B.

A *generalized hexagon of order* (s, t) is an incidence structure satisfying the following axioms:

(*i*) Any two distinct points are incident with at most one common line.

(*ii*) Any two distinct lines are incident with at most one common point.

(*iii*) Any line is incident with exactly $s + 1$ points and any point is incident with exactly $t + 1$ lines.

(*iv*) A smallest cycle consists of six points and six lines.

For descriptions of the classical generalized hexagon of order s, the interested reader is referred to [11, 28].

To date, projective planes of order s, generalized quadrangles of order s and generalized hexagons of order s have been obtained only when s is a prime power. For a description of the realizable parameters s and t for generalized quadrangles (hexagons) of order (s, t), the interested reader is referred to [11, Chapter 9].

We next define an *ovoid* of a finite generalized quadrangle (hexagon) Ω of order s to be a set S of $1+s^2$ $(1+s^3)$ points, any two being at distance 4 (6) in the incidence graph of Ω. Dually, a *spread* of a finite generalized quadrangle (hexagon) is a set of $1+s^2$ $(1+s^3)$ lines, any two being at distance 4 (6) in the incidence graph of Ω. Our interest in ovoids and spreads of these incidence structures will be revealed in the next subsection.

Theorem 2.3 ([11, Chapter 9]) *The symplectic generalized quadrangle of order s always has spreads.*

Our interest in incidence structures primarily revolves around the graphs that can be obtained from them, such as incidence graphs (already defined) and polarity graphs.

Let Γ be a bipartite graph with partite sets V_1 and V_2. A *polarity* ω on Γ is an involution of the automorphism group of Γ that interchanges V_1 and V_2; that is, $\omega(V_1) = V_2$ and $\omega(V_2) = V_1$.

Let $\Omega = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ be an incidence structure with a polarity ω (a polarity of the incidence graph). The *polarity graph*, denoted by Γ^ω, of Ω with respect to ω is the graph with vertex set $V(\Gamma) = \mathcal{P}$, and the following adjacency relation: $pp_1 \in E(\Gamma^\omega)$ if $p \neq p_1$ and $(p, \omega(p_1)) \in \mathcal{I}$. We call a point p an *absolute point* of the polarity ω if $(p, \omega(p)) \in \mathcal{I}$. The number of absolute points of ω is denoted by N_ω.

Next we need to know when a particular generalized polygon admits a polarity.

Theorem 2.4 ([11, Chapter 4]) *A projective plane of order s admits a polarity ω for every prime power s. Furthermore, $s+1 \leq N_\omega \leq s\sqrt{s}+1$.*

In Appendix A we describe a family of projective planes of order s and a polarity in that family with exactly $s+1$ absolute points.

Theorem 2.5 ([11, Chapter 7]) *The symplectic generalized quadrangle of order s admits a polarity ω if, and only if, $s = 2^{2\alpha+1}$, α being a natural number. Furthermore, $N_\omega = s^2+1$.*

In Appendix B we describe a polarity in the symplectic generalized quadrangle of order s.

Theorem 2.6 ([11, 12]) *A generalized hexagon of order s admits a polarity ω if, and only if, $s = 3^{2\alpha+1}$, α being a natural number. Furthermore, $N_\omega = s^3 + 1$.*

A description of such a polarity can be found in [16]; see also [44].

Finally, we state some relations between the corresponding incidence and polarity graphs of an incidence structure. The properties (*i*) and (*ii*) follow from the definition of Γ^ω, and were presented in [35, Theorem 1]. While the property (*iii*) seems to be taken for granted by some researchers, we could not find any reference to it. Therefore, here we provide a proof.

Theorem 2.7 *Let Ω be a generalized D-gon with a polarity ω, and Γ and Γ^ω the incidence and polarity graphs of Ω, respectively. Then the following assertions hold.*

(*i*) $d_{\Gamma^\omega}(p) = d_\Gamma(p) - 1$ *if p is an absolute point of ω, otherwise* $d_{\Gamma^\omega}(p) = d_\Gamma(p)$.

(*ii*) $|V(\Gamma^\omega)| = \frac{1}{2}V(\Gamma)$ *and* $|E(\Gamma^\omega)| = |E(\Gamma)| - N_\omega$.

(*iii*) *If* $\Delta(\Gamma) \geq 3$ *then* $D(\Gamma^\omega) = D(\Gamma) - 1$.

Proof. We first prove that $D(\Gamma^\omega) \leq D(\Gamma) - 1$. Let us take two vertices p_x and p_y in Γ^ω. If $D(\Gamma)$ is odd, then in Γ there exists a path of length at most $D(\Gamma) - 1$ between p_x and p_y, implying the existence in Γ^ω of a path of length at most $D(\Gamma) - 1$ between p_x and p_y. If instead $D(\Gamma)$ is even, then in Γ there exists a path of length at most $D(\Gamma) - 1$ between p_x and $\omega(p_y)$, therefore the assertion follows.

Next we prove that $D(\Gamma^\omega) > D(\Gamma) - 2$. To prove this it suffices to prove that $|\Gamma^\omega| > M_{\Delta,D-2}$. Note that

$$|\Gamma^\omega| = \sum_{k=0}^{D(\Gamma)-1} (\Delta - 1)^k$$

and that

$$M_{\Delta,D-2} = 2 \sum_{k=0}^{D(\Gamma)-3} (\Delta - 1)^k + (\Delta - 1)^{D(\Gamma)-2}$$

For $D(\Gamma) \geq 3$ and $\Delta \geq 3$, it is easy to prove by induction on $D(\Gamma)$ that

$$(\Delta-1)^{D(\Gamma)-2}-1 \geq \sum_{k=0}^{D(\Gamma)-3} (\Delta-1)^k$$

Therefore

$$\sum_{k=0}^{D(\Gamma)-2} (\Delta-1)^k + (\Delta-1)^{D(\Gamma)-2} - 1 \geq M_{\Delta,D-2}$$

Since $(\Delta-1)^{D(\Gamma)-1} > (\Delta-1)^{D(\Gamma)-2}-1$, we have

$$|\Gamma^\omega| > \sum_{k=0}^{D(\Gamma)-2} (\Delta-1)^k + (\Delta-1)^{D(\Gamma)-2} - 1$$

and the assertion follows. □

2.2 Vertex Duplication

Given a (Δ, D)-graph Γ, the technique of *vertex duplication* [18] consists of selecting a vertex x of Γ and adding a new vertex x' (the duplicate of x) such that $N(x') = N(x) \cup \{x\}$. The resulting graph Γd clearly has maximum degree $\Delta+1$ and diameter D. Consequently

$$N_{\Delta+1,D} \geq N_{\Delta,D} \tag{2}$$

In this regard, Delorme and Farhi [18] noted that if in a graph we have a set S of vertices with pairwise distance at least 3, then we can duplicate all the vertices in S, providing a better inequality than Inequality (2)

$$N_{\Delta+1,D} \geq N_{\Delta,D} + |S|.$$

So, the application of the technique of vertex duplication often reduces to the finding of sets of vertices at mutual distance at least 3. In this regard, if we apply the technique to incidence graphs of generalized polygons, ovoids and spreads constitute examples of these desired sets.

2.3 Graph Compounding

This technique was introduced in [6], and it has been applied many times to successfully produce large graphs [4, 5, 14, 42].

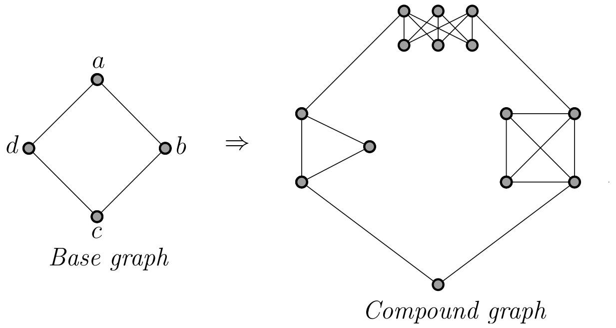

Figure 1: Compound graph.

Let $S = \{\Lambda_1, \Lambda_2, \ldots, \Lambda_k\}$ be a set of graphs. Each element of S is called a *source graph*, and consequently, S is called the *set of source graphs.* Let Γ be a graph, called the *base graph.* In addition, let $\hat{\Gamma}$ be a subgraph of Γ such that $E(\hat{\Gamma}) = \emptyset$, and $V(\hat{\Gamma})$ is formed by all those vertices of Γ to be replaced during the compounding process. We call $\hat{\Gamma}$ the *replaced graph.* Finally, let f be a mapping from $V(\hat{\Gamma})$ to S.

The compounding of S into Γ is denoted by $\Gamma(S)$ or by $\Gamma(\Lambda_1, \Lambda_2, \ldots, \Lambda_k)$. We define it by means of the following two steps:

Step 1: Every vertex $x \in V(\hat{\Gamma})$ is replaced by the graph $f(x) \in S$. The set of added vertices is denoted by $\hat{V}(S)$, that is, $\hat{V}(S) = \bigcup\limits_{x \in V(\hat{\Gamma})} V(f(x))$.

Step 2: The edges incident with $x \in V(\hat{\Gamma})$ are distributed among the vertices of $f(x)$. Note that this step introduces a certain amount of ambiguity. We always need to specify how this step is done.

It is easy to see that

$$|\Gamma(S)| = |\Gamma| + \sum_{x \in V(\hat{\Gamma})} |f(x)| - |\hat{\Gamma}|.$$

To exemplify this, see Figure 1, where $S = \{K_{3,3}, K_4, K_3\}$ and the base graph is C_4. The replaced graph has vertex set $V(\hat{\Gamma}) = \{a, b, d\}$ and edge set $E(\hat{\Gamma}) = \emptyset$, and the mapping is $f(a) = K_{3,3}$, $f(b) = K_4$, and $f(d) = K_3$.

2.4 Kronecker Products

The *Kronecker product* [17] of two bipartite graphs Γ (with partite sets A and B) and Γ' (with partite sets A' and B') has vertex set $(A \times A') \cup (B \times B')$, and $(a, a') \sim (b, b')$ if, and only if, $ab \in E(\Gamma)$ and $a'b' \in E(\Gamma')$. The resulting graph is bipartite, has diameter $\max\{D(\Gamma), D(\Gamma')\}$, order $|A||A'| + |B||B'|$ and maximum degree $\max\{\Delta_A \Delta_{A'}, \Delta_B \Delta_{B'}\}$, where Δ_X denotes the maximum degree of the vertices in X.

Given a bipartite graph $\Gamma = (A \cup B, E)$, the Kronecker product of Γ and its "opposite" $(B \cup A, E)$, denoted by $\otimes\Gamma$, has a polarity ω, which is defined by $\omega((a, b)) = (b, a)$. Then, to obtain large graphs of diameter $D(\Gamma) - 1$, we can consider the "polarity graph" of the Kronecker product of Γ and its opposite with respect to ω, denoted $(\otimes\Gamma)^\omega$.

2.5 Voltage Assignments and Voltage Graphs

Many of the largest known graphs of maximum degree $3 \leq \Delta \leq 16$ and diameter $2 \leq D \leq 10$, were constructed using voltage graphs [38, 39]. See [29] for a thorough treatment of voltage graphs, and [8, 9] for their applications in the construction of large graphs.

Let Γ be a finite, undirected graph, possibly with loops and multiple edges. We also allow *semi-edges*, which are edges with just one end-vertex and with the other end free. To facilitate the description of voltages, we think of the (undirected) edges of Γ that are not semi-edges as pairs of oppositely directed arcs. A semi-edge admits, by definition, just one direction (into its unique end vertex). So, we obtain the digraph Λ. The number of elements in the set $A(\Lambda)$ of all arcs of Λ is therefore twice the number of all edges of Γ minus the number of semi-edges. If e is an arc, then e^{-1} denotes the arc reverse to e; in the case of a semi-edge we set $e^{-1} = e$ by convention.

Let G be a finite group. A mapping $\alpha : A(\Lambda) \to G$ is called a *voltage assignment* if $\alpha(e^{-1}) = (\alpha(e))^{-1}$ for every arc $e \in A(\Lambda)$. Thus, a voltage assignment sends a pair of mutually reverse arcs onto a pair of mutually inverse elements of the group. Note that if e is a semi-edge, then the voltage condition means that $\alpha(e)$ has order 2 in G. The pair (Λ, α) is the *voltage*

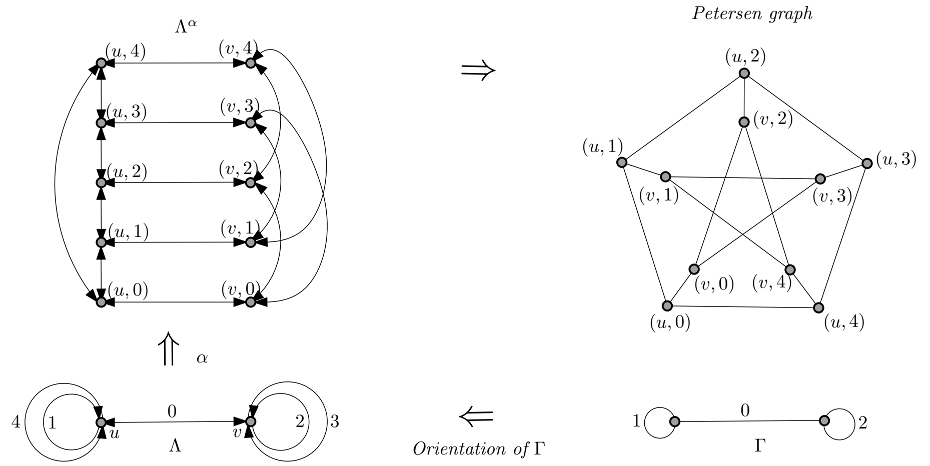

Figure 2: Petersen graph obtained by the voltage assignment technique. The voltages are taken from $G = \mathbb{Z}_5$.

graph, which determines a *lift* Λ^α of Λ as follows. The vertex set and the arc set of the lift are $V(\Lambda^\alpha) = V(\Lambda) \times G$ and $A(\Lambda^\alpha) = A(\Lambda) \times G$, respectively. In the lift, (e, g) is an arc from the vertex (u, g) to the vertex (v, h) if, and only if, e is an arc from u to v in Λ and $h = g\alpha(e)$. The lift itself is considered to be undirected, since (e, g) and $(e^{-1}, g\alpha(e))$ form a pair of mutually reverse arcs and therefore give rise to an undirected edge of Λ^α. See Figure 2.

Let $\pi : \Lambda^\alpha \to \Lambda$ be a function given by $\pi((e, g)) = e$ and $\pi((v, g)) = v$; π is called *natural projection*. This is the reason why Λ is often called a (regular) *quotient* of Λ^α. For any vertex v and any arc e of the quotient, the sets $\pi^{-1}(v)$ and $\pi^{-1}(e)$ are called *fibres* above v and e, respectively.

It is clear that the degree of a vertex v in Λ is inherited by all vertices in the fibre above v in Λ^α. This gives a trivial way to control vertex degrees in the lifts. To determine the diameter of the lift, it is sufficient to choose one vertex from each fibre and check all distances in the lift from the chosen vertices.

2.5.1 Cayley Graphs

Let G be a group, and let X be a set of generators of G such that

(i) if $x \in X$ then $x^{-1} \in X$, and

(ii) X does not contain the identity of G.

The elements of G form the set of vertices of the undirected *Cayley* graph $C(G, X)$. Given u, v in G, the edge $\{u, v\}$ in $C(G, X)$ exists if $u^{-1}v \in X$. For example, for any given n the complete graph K_n is isomorphic to the Cayley graph $C(\mathbb{Z}_n, \{1, 2, ..., n\})$.

Cayley graphs $C(G, X)$ can be considered as a special class of voltage graphs, as all Cayley graphs can be obtained as lifts of *bouquets*, that is, a single-vertex graph with s semi-edges and l loops, and denoted by $B(s, l)$.

2.5.2 Semi-direct Products

Given two groups X and Y and a homomorphism $\psi : Y \to \mathrm{Aut}(X)$, the semi-direct product gives a group structure to the set $X \times Y$, where the multiplication in the first coordinate is twisted by ψ. The automorphism assigned to y by ψ will be denoted ψ_y. The resulting multiplication rule in the group is $(x, y)(x', y') = (x\psi_y(x'), yy')$. This semi-direct product is denoted $X \rtimes_\psi Y$. Here we note that in the natural identification $x \in X \to (x, 1) \in X \rtimes_\psi Y$ and $y \in Y \to (1, y) \in X \rtimes_\psi Y$, the product yxy^{-1} corresponds to $(1, y)(x, 1)(1, y^{-1}) = (\psi_y(x), 1)$, and hence the conjugation of x by y corresponds to applying the automorphism ψ_y to x.

2.5.3 Semi-direct products of the form $\mathbb{Z}_m \rtimes_r \mathbb{Z}_n$

All the voltage graphs presented in this paper are lifts of bouquets with voltages from semi-direct products of cyclic groups, and therefore are all isomorphic to Cayley graphs. In precedent works [21, 38, 45], semi-direct products of cyclic groups were used with great success. One possible explanation for this phenomenon is that these groups can have large automorphism groups. Nevertheless, a complete understanding of this phenomenon is not available yet.

To define semi-direct products of cyclic groups, consider, for example, the cyclic groups $\mathbb{Z}_m$

and $\mathbb{Z}_n$, such that $\gcd(\phi(m), n) > 1$, where $\phi(m)$ is the so-called *Euler's totient function*[1]. Let $\psi(x) = r^x \pmod{m}$, where $r^n \equiv 1 \pmod{m}$. Thus, the group operation is $(x_1, y_1)(x_2, y_2) = (x_1 + r^{y_1}x_2, y_1 + y_2)$. The resulting semi-direct product is denoted by $Z_m \rtimes_r Z_n$. It is easy to confirm that $yxy^{-1} = \psi_y(x) = (0, y)(x, 0)(0, -y) = (0 + r^y x, y)(0, -y) = (r^y x, 0) \equiv r^y x \pmod{m}$.

2.6 Compound Graphs $B(T_n)$

Next we define a family of compound graphs obtained by Gómez and Fiol [26]. These graphs are denoted by $B(T_n)$[2] and constitute some of largest known graphs of diameters 4 and 6 as we will see below. To define such graphs we use the definition of graph compounding provided above.

Source Graphs: $\Lambda_1 = B$, where $B = (V_0 \cup V_1, E)$ is any bipartite graph with even diameter and $|V_0| = |V_1|$.

Base Graph: A *tournament* $\Gamma = T_n$ of order n, that is, a digraph obtained by assigning a direction to each edge in a complete graph. See Figure 4 for some examples.

Replaced Graph: $\hat{\Gamma} = (V(T_n), \emptyset)$.

Mapping f:

$$f(v) = 1 \qquad \text{for all } v \in V(T_n)$$

Specifying Step 2: Let (t, s) be an arc in T_n, and let B^t and B^s be the copies of B substituting vertices t and s, respectively.

Vertex $v \in V(B^t)$ that belongs to V_0 (respectively, V_1) is denoted by $(v, 0, t)$ (respectively, $(v, 1, t)$).

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[1] $\phi(m)$ is defined as the number of positive integers smaller than m and relatively prime to m.

[2] The notation of this family is apparently bizarre. Following our notation, which is in line with traditional practices, the family $B(T_n)$ should have been denoted by $T_n(B)$ because we are replacing vertices of T_n with copies of B. However, to keep consistency with the original source we have maintained the author's notation.

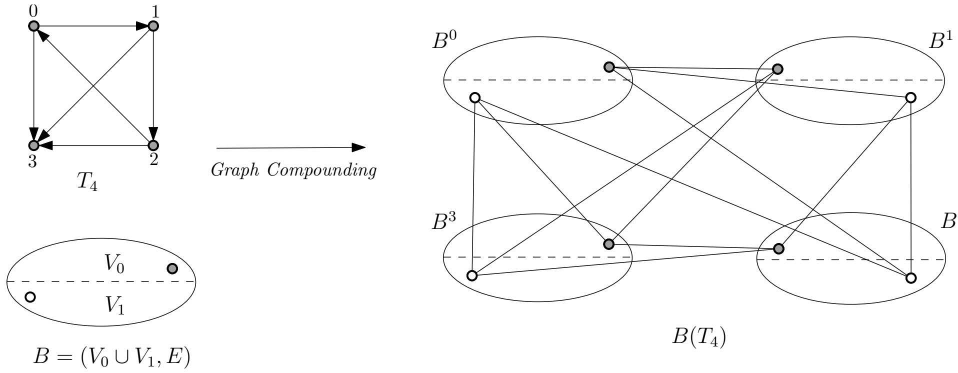


Figure 3: Basic Configuration of a $B(T_4)$. For the arc $(0,1)$ of T_4 we select the partite set V_0 of B^0, for the arc $(1,2)$ of T_4 we select the partite set V_1 of B^1, for the arc $(2,3)$ of T_4 we select the partite set V_0 of B^2, for the arc $(0,3)$ of T_4 we select the partite set V_1 of B^0, for the arc $(1,3)$ of T_4 we select the partite set V_0 of B^1, and for the arc $(2,0)$ of T_4 we select the partite set V_1 of B^2.

For each arc (t,s) of T_n we select a partite set V_i^t of B^t. This selection is done in such a way that the maximum degree of $B(T_n)$ is as small as possible; see Figure 3. Then, for each vertex $(v,i,t) \in V(B^t)$, two vertices $(u,0,s)$ and $(w,1,s)$ in $V(B^s)$ are chosen (not selected previously). Then the new edges $\{(v,i,t),(u,0,s)\}$ and $\{(v,i,t),(w,1,s)\}$ are introduced.

Resulting Graph: $B(T_n)$

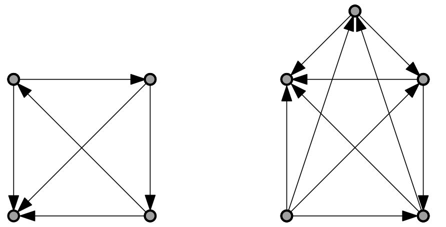
Features:

(i) **Maximum Degree:** $\Delta(B)+ p(T_n)$, where $p(T_n) = \max\limits_{t \in V(T_n)} (d^-(t) + 2\left\lceil \frac{d^+(t)}{2} \right\rceil)$.

(ii) **Diameter:** $D(B)$.

(iii) **Order:** $|T_n| \times |B|$.

Observation 2.1 *Let $B = (V_0 \cup V_1, E)$ be a bipartite graph. If $S \subset V_0$ is a set of vertices at pairwise distance at least 3, then, for any vertex $t \in T_n$, S will also be a set of vertices at pairwise distance at least 3 in the partite set V_0 of B^t after the formation of $B(T_n)$.*



T_4 with $p(T_4) = 3$ T_5 with $p(T_5) = 4$

Figure 4: Some tournaments with minimum $p(T_n)$: a T_4 and a T_5.

2.7 Compound Graphs $B_0\Theta_4B_1$

Next we present a family of compound graphs obtained by Gómez and Miller [27]: the compound graphs $B_0\Theta_4B_1$.

Definition 2.1 (Compound graph $B_0\Theta_4B_1$ [27]) *Let $B_i = (V_{i0} \cup V_{i1}, E)$ be a regular bipartite graph with even diameter, where $i = 0, 1$. Then, a compound graph $B_0\Theta_4B_1$ is defined as follows:*

- *(i) Take 4 sets A_0, A_1, A_2 and A_3 of $\frac{|B_1|}{2}$ copies of B_0 each, that is, $|A_k| = \frac{|B_1|}{2}$ with $k = 0, 1, 2, 3$.*
- *(ii) Take 4 sets F_0, F_1, F_2 and F_3 of $\frac{|B_0|}{2}$ copies of B_1 each, that is, $|F_k| = \frac{|B_0|}{2}$ with $k = 0, 1, 2, 3$.*
- *(ii) In the set A_k, the vertex s of the stable set V_{0j} from the copy t of B_0 (denoted B_0^{tk}) is denoted by $(s, 0, j, k, t)$, where $0 \leq s \leq \frac{|B_0|}{2} - 1$, $0 \leq t \leq \frac{|B_1|}{2} - 1$, $j = 0, 1$ and $k = 0, 1, 2, 3$.*
- *(iii) In the set F_k, the vertex t of the stable set V_{1j} from the copy s of B_1 (denoted B_1^{sk}) is denoted by $(t, 1, j, k, s)$, where $0 \leq s \leq \frac{|B_0|}{2} - 1$, $0 \leq t \leq \frac{|B_1|}{2} - 1$, $j = 0, 1$ and $k = 0, 1, 2, 3$.*
- *(iv) The adjacency rules are given by Figure 5.*

Theorem 2.8 (Compound graph $B_0\Theta_4B_1$ [27]) *The graph $B_0\Theta_4B_1$ presents the following properties:*

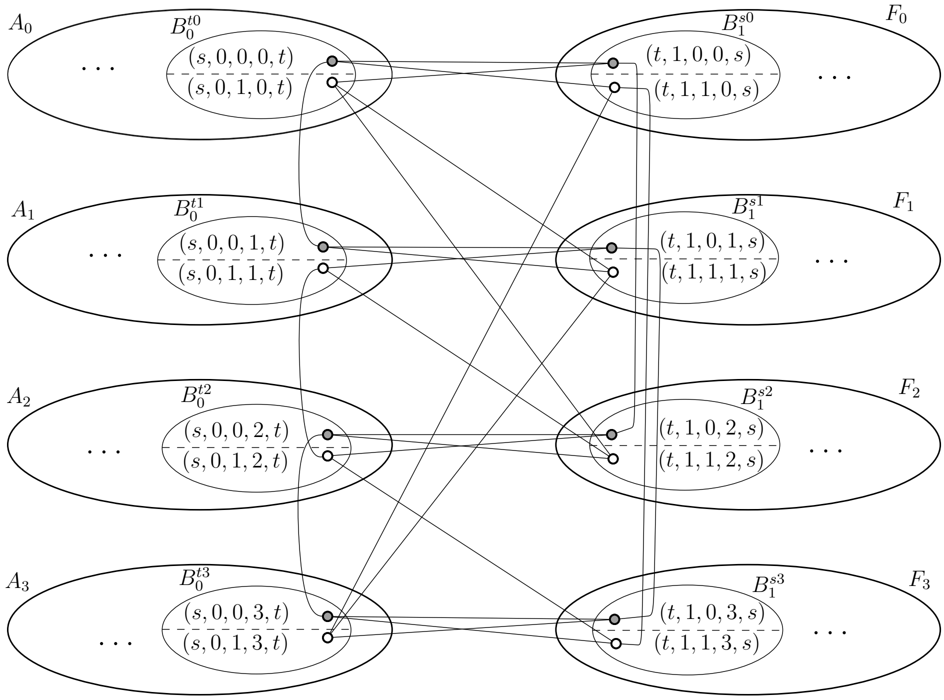


Figure 5: Configuration of the adjacency rules in $B_0\Theta_4B_1$.

(i) $\Delta(B_0\Theta_4B_1) = \max(\Delta(B_0)+3, \Delta(B_1)+3)$.

(ii) $D(B_0\Theta_4B_1) \le D(B_0)+D(B_1)$.

(iii) $|B_0\Theta_4B_1| = 4|B_0| \times |B_1|$.

3 New Largest Known Graphs

In this section we present a complete description of all the new largest known graphs of maximum degree $17 \le \Delta \le 20$ and diameter $2 \le D \le 10$. Our purpose is to give a practical insight into the applications of the methods described in the previous section, so other graphs can be reproduced to model LSNs in various applications. In each subsection we show a table listing the orders of the largest known graphs for the corresponding diameter D and maximum degree $17 \le \Delta \le 20$. The percentage of the Moore bound of each of the corresponding orders is in the column $\%M_{\Delta,D}$.

Δ	Order	$\%M_{\Delta,D}$	Graph
17	**274**	95.13	EFH-Brown Graph
18	**307**	95.04	Brown Graph
19	**338**	93.88	MMS Graph
20	**381**	95.48	Brown Graph

Table 1: Orders of the largest known graphs of diameter 2 and maximum degrees $17 \leq \Delta \leq 20$.

Additional tables detailing the group and elements for the voltage graphs are given in Appendix C.

3.1 Graphs of Diameter 2

For diameter 2, Brown [10] constructed a graph Γ of maximum degree Δ and order $1 + (\Delta - 1) + (\Delta - 1)^2$, for each Δ such that $\Delta - 1$ is a prime power. The Brown graph (isomorphic to the Erdős-Rényi graph [23]) is a polarity graph of a projective plane of order $\Delta - 1$, whose polarity (defined in Appendix A) has Δ absolute points. Therefore, the Brown graph has precisely Δ vertices of degree $\Delta - 1$ (the absolute points of the polarity) and $(\Delta - 1)^2$ vertices of degree Δ.

As projective planes of order $\Delta - 1$ admit a polarity for every prime power $\Delta - 1$ (by Theorem 2.4), for maximum degrees 18 and 20 the largest known graphs are the Brown graphs for these values, whose orders are 307 and 381, respectively.

Erdős, Fajtlowicz and Hoffman [22] noted that if $\mathcal{F}$ is a field whose order is a power of 2 then Brown's construction can be slightly improved by producing a graph (called EFH-Brown graph) of order $2 + (\Delta - 1) + (\Delta - 1)^2$. Therefore, the largest graph of maximum degree 17 is a EFH-Brown graph of order 274.

The largest graph of maximum degree 19 is provided by a McKay-Miller-Širáň graph (MMS graph). McKay-Miller-Širáň graphs [31, 32, 39] constitute an infinite family of regular graphs of degree $\Delta = \frac{3q-\gamma}{2}$ and order $\frac{8}{9}\left(\Delta + \frac{\gamma}{2}\right)^2$, whenever q is a prime power congruent with γ (mod 4), and $\gamma \in \{-1, 0, 1\}$.

Δ	Order	$\%M_{\Delta,D}$	Graph
17	**1,610**	34.69	$(\otimes Q_3)^{\omega}d$
18	**1,620**	29.31	$(\otimes Q_3)^{\omega}d^2$
19	**1,638**	25.13	Cayley
20	**1,958**	25.69	Cayley

Table 2: Orders of the largest known graphs of diameter 3 and maximum degrees $17 \leq \Delta \leq 20$.

3.2 Graphs of Diameter 3

To obtain large graphs of maximum degrees 17 and 18, we duplicate some vertices of the polarity graph $(\otimes Q_3)^{\omega}$ of the Kronecker product of Q_3 and its "opposite". The graph $(\otimes Q_3)^{\omega}$ has order 1600 and is the largest known graph of maximum degree 16 and diameter 3.

Consider $Q_{\Delta-1}$, with partite sets $P = \{p_1, \dots, p_m\}$ and $L = \{l_1, \dots, l_m\}$, and a spread $S = \{l_{j1}, \dots, l_{j((\Delta-1)^2+1)}\}$ (see Theorem 2.3). It is not difficult to see that in $\otimes Q_{\Delta-1}$, for each i the vertices $(p_i, l_{j1}), \dots, (p_i, l_{j((\Delta-1)^2+1)})$ are also at pairwise distance 4. Let $V((\otimes Q_{\Delta-1})^{\omega}) = W_1 \cup \dots \cup W_m$, where $W_i = \{(p_i, l_1), \dots, (p_i, l_m)\}$. Then, in $(\otimes Q_{\Delta-1})^{\omega}$ for each i the vertices $(p_i, l_{j1}), \dots, (p_i, l_{j((\Delta-1)^2+1)})$ are at pairwise distance 3. As a result, by using the duplication technique, we can select a subset W_i, say W_1, and duplicate the vertices $(p_1, l_{j1}), \dots, (p_1, l_{j((\Delta-1)^2+1)})$ obtaining a new graph $(\otimes Q_{\Delta-1})^{\omega}d$ of maximum degree Δ^2+1, diameter 3, and order $(\Delta-1)^2 + 1 + |(\otimes Q_{\Delta-1})^{\omega}|$. If instead we select k subsets W_i, say $W_1, \dots, W_k$, we can then duplicate the vertices $(p_i, l_{j1}), \dots, (p_i, l_{j((\Delta-1)^2+1)})$ in each of these W_i, obtaining a new graph $(\otimes Q_{\Delta-1})^{\omega}d^k$ of maximum degree $\Delta^2 + k$, diameter 3, and order $k((\Delta-1)^2+1) + |(\otimes Q_{\Delta-1})^{\omega}|$.

In Q_3, $|S| = 10$, and thus, the largest known graphs of maximum degree 17 and 18 and diameter 3 are the graphs $(\otimes Q_{\Delta-1})^{\omega}d$ and $(\otimes Q_{\Delta-1})^{\omega}d^2$ of orders 1610 and 1620, respectively.

The largest known graphs of maximum degree 19 and 20 and diameter 3 are Cayley graphs of semi-direct products of cyclic groups of the form $\mathbb{Z}_m \rtimes_r \mathbb{Z}_n$. The groups and voltages are in Table 10 in Appendix C.

Δ	Order	$\%M_{\Delta,D}$	Graph
17	**19,040**	25.63	$Q_{13}(T_4)$
18	**23,800**	25.32	$Q_{13}(T_5)$
19	**23,970**	20.43	$Q_{13}(T_5)d$
20	**34,952**	24.13	$Q_{16}(T_4)$

Table 3: Orders of the largest known graphs of diameter 4 and maximum degrees $17 \leq \Delta \leq 20$.

3.3 Graphs of Diameter 4

The largest known graphs of maximum degrees 17, 18 and 20 are provided by the family of compound graphs $B(T_n)$, using $Q_{\Delta-1}$ (maximal bipartite graphs of diameter 4 for their corresponding degrees [37]) as source graphs and the tournaments depicted in Figure 4 as base graphs. Note that we select tournaments T_n with minimum $p(T_n)$ for the corresponding n.

It is also not difficult to see that $p(T_6) \geq 6$. In fact, suppose, by way of contradiction, that there is a T_6 with $p(T_6) = 5$. Then each vertex $u \in V(T_6)$ must have either $d^+(u) = 2$ and $d^-(x) = 3$ or $d^+(u) = 4$ and $d^-(x) = 1$. But, in this case, for any distribution of vertices we have that $\sum_{x \in V(T_6)} d^+(x) \neq \sum_{x \in V(T_6)} d^-(x)$, which is a contradiction. As a graph $\Delta(Q_{13}(T_6)) \geq 20$, we cannot obtain a $B(T_n)$ graph of maximum degree 19 and diameter 4 with larger order than $Q_{13}(T_5)$. See Table 3.

Therefore, to obtain a large graph of maximum degree 19 and diameter 4, we duplicate some vertices of $Q_{13}(T_5)$. From Theorem 2.3 it follows that in Q_{13} we can find a spread S. So, by Observation 1, duplicating the vertices of S we obtain a graph $Q_{13}(T_5)d$ of maximum degree 19, diameter 4 and order $|Q_{13}(T_5)| + |S|$, where $|S| = 170$.

3.4 Graphs of Diameter 5

The largest known graphs of maximum degrees 17–20 are provided by Cayley graphs of semidirect products of cyclic groups of the form $\mathbb{Z}_m \rtimes_r \mathbb{Z}_n$. The groups and voltages are in Table 11 in Appendix C.

Δ	Order	$\%M_{\Delta,D}$	Graph
17	**133,144**	11.20	Cayley
18	**171,828**	10.75	Cayley
19	**221,676**	10.49	Cayley
20	**281,820**	10.24	Cayley

Table 4: Orders of the largest known graphs of diameter 5 and maximum degrees $17 \leq \Delta \leq 20$.

3.5 Graphs of Diameter 6

The largest known graphs of maximum degrees 17,18 and 20 are provided by the family of compound graphs $B(T_n)$, using $H_{\Delta-1}$ (maximal bipartite graphs of diameter 6 for their corresponding degrees [37]) as source graphs and the tournaments depicted in Figure 4 as base graphs. As in the case of diameter 4, we cannot obtain good $B(T_n)$ graphs for maximum degree 19.

We first observe that in a $H_{\Delta-1}$ we can select a set S of $\Delta(\Delta-1)^2+1$ points at pairwise distance at least 4. Indeed, select a point $p_0 \in V(H_{\Delta-1})$, and consider the set $N_k(p_0)$ of elements at distance k from p_0. For each line $l_i \in N_3(p_0)$, select a point $p_i \in N(l_i) \cap N_4(p)$, for $1 \leq i \leq \Delta(\Delta-1)^2$. The girth of $H_{\Delta-1}$ is 12, thus, the set $S = \{p, p_1, p_2, \ldots, p_{\Delta(\Delta-1)^2}\}$ is formed by points at mutual distance at least 4.

To obtain a large graph of maximum degree 19 and diameter 6, by the previous paragraph and Observation 1, we can find a set S of $\Delta(\Delta-1)^2+1$ points at pairwise distance at least 4 in $H_{13}(T_5)$. Duplicating the vertices of S, we obtain a graph $H_{13}(T_5)d$ of maximum degree 19, diameter 6 and order $|H_{13}(T_5)| + 14 \times 13^2 + 1$.

3.6 Graphs of Diameters 7, 8 and 9

The largest known graphs of maximum degrees 17–20 and diameters 7, 8 and 9 are provided by Cayley graphs of semi-direct products of cyclic groups of the form $\mathbb{Z}_m \rtimes_r \mathbb{Z}_n$. The groups and voltages are in Tables 12, 13 and 14 in Appendix C.

Δ	Order	$\%M_{\Delta,D}$	Graph
17	**3,217,872**	16.92	$H_{13}(T_4)$
18	**4,022,340**	14.81	$H_{13}(T_5)$
19	**4,024,707**	10.58	$H_{13}(T_5)d$
20	**8,947,848**	17.11	$H_{16}(T_4)$

Table 5: Orders of the largest known graphs of diameter 6 and maximum degrees $17 \leq \Delta \leq 20$.

Δ	Order	$\%M_{\Delta,D}$	Graph
17	**18,495,162**	6.07	Cayley
18	**26,515,120**	5.74	Cayley
19	**39,123,116**	5.71	Cayley
20	**55,625,185**	5.6	Cayley

Table 6: Orders of the largest known graphs of diameter 7 and maximum degrees $17 \leq \Delta \leq 20$.

Δ	Order	$\%M_{\Delta,D}$	Graph
17	**220,990,700**	4.54	Cayley
18	**323,037,476**	4.11	Cayley
19	**501,001,000**	4.06	Cayley
20	**762,374,779**	4.04	Cayley

Table 7: Orders of the largest known graphs of diameter 8 and maximum degrees $17 \leq \Delta \leq 20$.

Δ	Order	$\%M_{\Delta,D}$	Graph
17	**3,372,648,954**	4.33	Cayley
18	**5,768,971,167**	4.32	Cayley
19	**8,855,580,344**	3.99	Cayley
20	**12,951,451,931**	3.61	Cayley

Table 8: Orders of the largest known graphs of diameter 9 and maximum degrees $17 \leq \Delta \leq 20$.

Δ	Order	$\%M_{\Delta,D}$	Graph
17	**15,317,070,720**	1.22	$Q_{13}\Theta_4 H_{13}$
18	**16,659,077,632**	0.73	$Q_{13}d\Theta_4 H_{13}d$
19	**18,155,097,232**	0.45	$Q^*_{13}d\Theta_4 H^*_{13}d$
20	**78,186,295,824**	1.14	$Q_{16}\Theta_4 H_{16}$

Table 9: Orders of the largest known graphs of diameter 10 and maximum degrees $17 \leq \Delta \leq 20$.

3.7 Graphs of Diameter 10

The largest graphs of maximum degrees 17 to 20 and diameter 10 are members of the family $B_0\Theta_4 B_1$ of compound graphs. These graphs are obtained using the largest known bipartite graphs of diameter 4 for B_0 and the largest known bipartite graphs of diameter 6 for B_1. That is, for maximum degrees 17 and 20, the graphs B_0 are the bipartite graphs Q_{13} and Q_{16} respectively, while for maximum degrees 18 and 19 we use the largest bipartite graphs of diameter 4 and maximum degree 15 and 16, respectively ($Q_{13}d$ and $Q^*_{13}d$ which are obtained by applying vertex duplication to Q_{13} [19]). For maximum degrees 17 and 20, the graphs B_1 are the bipartite graphs H_{13} and H_{16} respectively, while for maximum degrees 18 and 19 we use the largest bipartite graphs of diameter 6 and maximum degrees 15 and 16 respectively ($H_{13}d$ and $H^*_{13}d$ which are obtained by applying vertex duplication to H_{13} [19]).

4 Summary

The graphs constructed in this paper provide lower bounds for $N_{\Delta,D}$ whenever $17 \leq \Delta \leq 20$ and $2 \leq D \leq 10$. These graphs are benchmarks for LSNs of given maximum degree and diameter lying on the aforementioned intervals. More importantly, this paper can be regarded as a manual to the state-of-the-art methodology used for constructing large graphs of bounded degree and small diameter, summarizing and demonstrating the best known techniques that were developed over a period of five decades.

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A A family of projective planes of order s

Description of this family was taken from [25, pp. 80–81].

Let $\mathcal{F}$ be the finite field of order s, s a prime power, and $\mathcal{V}$ a three-dimensional vector space over $\mathcal{F}$. The points of a projective plane I_s are the one-dimensional subspaces of $\mathcal{V}$, and the lines of I_s are the two-dimensional subspaces of $\mathcal{V}$. The point p is incident with the line l if, and only if, $p \in l$. Clearly, a point can be represented by a non-zero vector of $\mathcal{V}$, while given a non-zero vector $\vec{v}$ in $\mathcal{V}$, a line l can be represented by the set $l = \{\vec{x} \in \mathcal{V} | \vec{x} \cdot \vec{v} = 0\}$, where $\cdot$ denotes the dot product of two vectors.

A.1 A polarity with $s+1$ absolute points

Then a polarity ω of I_s with exactly $s+1$ absolute points can be defined as follows. To the point represented by the vector $\vec{v}$, ω associates the line $\{\vec{x} \in \mathcal{V} | \vec{x} \cdot \vec{v} = 0\}$; see [18].

B The symplectic generalized quadrangle of order s

Descriptions of this family can be found in [25, pp. 80–81], in [11, Chapter 9] and in [41]. The description presented here is mainly taken from [41].

Let $\mathcal{F}$ be the finite field of order s, s a prime power, and $\mathcal{V}$ a four-dimensional vector space over $\mathcal{F}$. The *projective space* $PG(3, s)$ is formed by the one-dimensional subspaces of $\mathcal{V}$ (points), the two-dimensional subspaces of $\mathcal{V}$ (lines) and the three-dimensional subspaces of $\mathcal{V}$ (planes).

Let $\vec{u} = (u_0, u_1, u_2, u_3)^T$ and $\vec{v} = (v_0, v_1, v_2, v_3)^T$ be two distinct vectors determining a line l of $PG(3, s)$, where $(u_0, u_1, u_2, u_3)^T$ denotes the transpose of (u_0, u_1, u_2, u_3). We consider vectors with *homogeneous coordinates*, that is, for any non-zero scalar $\alpha \in \mathcal{F}$, all vectors

$(\alpha u_0, \alpha u_1, \ldots, \alpha u_n)$ denote the same vector. Using the coordinates of these vectors we can define the *Plücker coordinates* [43] of l: $p_{ij} = \begin{vmatrix} u_i & u_j \\ v_i & v_j \end{vmatrix}$, where $0 \leq i < j \leq 3$ and $|\cdot|$ denotes the determinant. That is, $l = (p_{01}, p_{23}, p_{02}, p_{31}, p_{03}, p_{12})^T$. For more information and applications of Plücker coordinates, refer to [46].

It can be seen that $p_{01}p_{23} + p_{02}p_{31} + p_{03}p_{12} = 0$.

The vector $\vec{z} = (\mathbb{Z}_0, \mathbb{Z}_1, \mathbb{Z}_2, \mathbb{Z}_3)^T$ is on $l = (p_{01}, p_{23}, p_{02}, p_{31}, p_{03}, p_{12})^T$ if, and only if,

$$\begin{pmatrix} p_{12} & -p_{02} & p_{01} & 0 \\ -p_{31} & -p_{03} & 0 & p_{01} \\ p_{23} & 0 & -p_{03} & p_{02} \\ 0 & p_{23} & p_{31} & p_{12} \end{pmatrix} \begin{pmatrix} \mathbb{Z}_0 \\ \mathbb{Z}_1 \\ \mathbb{Z}_2 \\ \mathbb{Z}_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Denote by $\langle \vec{u}, \vec{v} \rangle$ the space spanned by the vectors $\vec{u}$ and $\vec{v}$. Then, we have that the correspondence $l = \langle \vec{u}, \vec{v} \rangle \leftrightarrow \langle (p_{01}, p_{23}, p_{02}, p_{31}, p_{03}, p_{12})^T \rangle$ is a bijection from the set of lines of $PG(3, s)$ to the set of points of the Klein quadric $\mathcal{Q}$.

The *Klein quadric* $\mathcal{Q}$ is the set of points $\vec{u} = (u_0, u_1, u_2, u_3, u_4, u_5)^T$, such that $u_0u_1 + u_2u_3 + u_4u_5 = 0$.

Let $H = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$ and $\vec{u}^{\perp} = \{\vec{v} \in \mathcal{V} | \vec{u}^T H \vec{v}\}$.

We say that a line $l = \langle \vec{u}, \vec{v} \rangle$ is *totally isotropic* (or *self-conjugate*) if $\vec{u} \neq \vec{v} \in \vec{u}^{\perp}$. In terms of Plücker coordinates, a line $l = (p_{01}, p_{23}, p_{02}, p_{31}, p_{03}, p_{12})^T$ is totally isotropic if $p_{01} = -p_{23}$.

To construct the symplectic generalized quadrangle $W(s)$, take the points of $PG(3, s)$ and the totally isotropic lines of $PG(3, s)$.

B.1 A polarity in $W(s)$

By Theorem 2.5, $s = 2^{2\alpha+1}$ for α a positive integer. Let $\sigma = 2^{\alpha+1}$ such that the map $x \mapsto x^{\sigma}$ is an automorphism of $\mathcal{F}$. Then a polarity ω of $W(s)$ from the points of $PG(3, s)$ to the totally

Order	Degree	Diameter	Group			$\%M_{\Delta,D}$	Quotient
			m	n	r		
1,638	19	3	91	18	3	25.13	$B(1,9)$
Voltages	[(0,9)|(77,8)(37,2)(75,2)(20,16)(75,3)(28,1) (17,8)(54,8)(67,8)]						
1,958	20	3	89	22	2	25.69	$B(0,10)$
Voltages	[(61,2)(0,13)(54,8)(88,15)(22,9)(19,5) (6,14)(70,12)(86,18)(14,13)]						

Table 10: Cayley graphs of diameter 3.

isotropic lines given by Plücker coordinates can be defined as follows.

$$\omega(u_0,u_1,u_2,u_3) = (a^{\frac{\sigma}{2}}, a^{\frac{\sigma}{2}}, u_0^{\sigma}, u_1^{\sigma}, u_2^{\sigma}, u_3^{\sigma})^T$$

where $a = u_0u_1 + u_2u_3$.

It is worth noting that the absolute points of ω form an ovoid. When $s = 2^{2\alpha+1}$ for $\alpha \geq 1$ this ovoid is called a *Tits ovoid.*

C List of generators for the Cayley graphs

In Tables 10, 11, 12, 13 and 14 we present all the Cayley graphs obtained as voltage graphs of groups of the form $\mathbb{Z}_m \rtimes_r \mathbb{Z}_n$, and their specifications.

For all Cayley graphs the quotients are $B(s,l)$. When using bouquets $B(1,l)$, the list of voltages has the form $[(a_0,b_0)|(a_1,b_1)\dots(a_l,b_l)]$, where (a_0,b_0) is the voltage on the semi-edge and $(a_1,b_1)\dots(a_l,b_l)$ are the voltages on the loops. In the case of $B(0,l)$ the list has the form $[(a_1,b_1)(a_2,b_2)\dots(a_l,b_l)]$. Furthermore, recall that the column labelled $\%M_{\Delta,D}$ represents the percentage of the Moore bound reached with the order of the corresponding graphs.

<table>
<tr><th rowspan="2">Order</th><th rowspan="2">Degree</th><th rowspan="2">Diameter</th><th colspan="3">Group</th><th rowspan="2">$\%M_{\Delta,D}$</th><th rowspan="2">Quotient</th></tr>
<tr><th>m</th><th>n</th><th>r</th></tr>
<tr><td>133,144</td><td>17</td><td>5</td><td>1513</td><td>88</td><td>38</td><td>11.20</td><td>$B(1,8)$</td></tr>
<tr><td>Voltages</td><td colspan="7">[(578,44)|(643,57)(1502,62)(1277,50)(788,11)(706,53)(1116,83)
(627,33)(156,69)]</td></tr>
<tr><td>171,828</td><td>18</td><td>5</td><td>4773</td><td>36</td><td>350</td><td>10.75</td><td>$B(0,9)$</td></tr>
<tr><td>Voltages</td><td colspan="7">[(1455,20)(2948,27)(110,3)(4747,2)(3589,33)(3991,12)
(693,14)(1881,22)(3269,22)]</td></tr>
<tr><td>221,676</td><td>19</td><td>5</td><td>2639</td><td>84</td><td>3</td><td>10.49</td><td>$B(1,9)$</td></tr>
<tr><td>Voltages</td><td colspan="7">[(2548,42)|(2042,13)(2076,5)(2199,39)(2006,82)(1687,48)(823,51)
(187,41)(2059,69)(1695,48)]</td></tr>
<tr><td>281,820</td><td>20</td><td>5</td><td>4697</td><td>60</td><td>26</td><td>10.24</td><td>$B(0,10)$</td></tr>
<tr><td>Voltages</td><td colspan="7">[(1879,20)(649,42)(1942,4)(854,47)(1230,32)(1297,17)
(1136,14)(2032,4)(86,31)(2416,12)]</td></tr>
</table>

Table 11: Cayley graphs of diameter 5.

Order	Degree	Diameter	Group			$\%M_{\Delta,D}$	Quotient
			m	n	r		
18,495,162	17	7	48929	378	355	6.07	$B(1,8)$
Voltages	[(32883,189)|(29263,200)(7577,313)(6909,306)(44237,217)(33783,359)(19022,214)(37522,297)(12569,360)]						
26,515,120	18	7	331439	80	2250	5.74	$B(0,9)$
Voltages	[(108706,23)(8199,65)(23558,43)(85677,62)(119754,68)(120782,24)(81000,36)(30171,42)(254572,41)]						
39,123,116	19	7	315509	124	1772	5.71	$B(1,9)$
Voltages	[(214308,62)|(212460,95)(280631,112)(26848,40)(169873,1)(128509,13)(76254,77)(248780,25)(227670,80)(62130,26)]						
55,625,185	20	7	1589291	35	5449	5.60	$B(0,10)$
Voltages	[(275698,27)(812475,33)(927322,15)(630459,26)(798625,32)(648259,11)(598829,8)(1147158,12)(381209,2)(554546,4)]						

Table 12: Cayley graphs of diameter 7

<table>
<tr><th rowspan="2">Order</th><th rowspan="2">Degree</th><th rowspan="2">Diameter</th><th colspan="3">Group</th><th rowspan="2">$\%M_{\Delta,D}$</th><th rowspan="2">Quotient</th></tr>
<tr><th>m</th><th>n</th><th>r</th></tr>
<tr><td>220,990,700</td><td>17</td><td>8</td><td>315701</td><td>700</td><td>130</td><td>4.54</td><td>$B(1,8)$</td></tr>
<tr><td>Voltages</td><td colspan="7">[(258894,350)|(158756,591)(263670,396)(281566,692)(7016,133)(233281,561)(123425,228)(199992,95)(231557,676)]</td></tr>
<tr><td>323,037,476</td><td>18</td><td>8</td><td>352661</td><td>916</td><td>431</td><td>4.11</td><td>$B(0,9)$</td></tr>
<tr><td>Voltages</td><td colspan="7">[(326489,128)(216493,578)(189007,734)(170512,781)(64411,620)(245489,40)(42767,761)(200603,46)(217125,897)]</td></tr>
<tr><td>501,001,000</td><td>19</td><td>8</td><td>501001</td><td>1000</td><td>949</td><td>4.06</td><td>$B(1,9)$</td></tr>
<tr><td>Voltages</td><td colspan="7">[(269063,500)|(315744,297)(52648,189)(377907,838)(422347,633)(477962,935)(164749,791)(342022,141)(126534,147)(217512,716)]</td></tr>
<tr><td>762,374,779</td><td>20</td><td>8</td><td>468577</td><td>1627</td><td>37</td><td>4.04</td><td>$B(0,10)$</td></tr>
<tr><td>Voltages</td><td colspan="7">[(204249,615)(369501,52)(41539,724)(36943,178)(72436,196)(161179,855)(140645,229)(12036,258)(119969,199)(344260,596)]</td></tr>
</table>

Table 13: Cayley graphs of diameter 8.

<table>
<tr><th rowspan="2">Order</th><th rowspan="2">Degree</th><th rowspan="2">Diameter</th><th colspan="3">Group</th><th rowspan="2">$\%M_{\Delta,D}$</th><th rowspan="2">Quotient</th></tr>
<tr><th>m</th><th>n</th><th>r</th></tr>
<tr><td>3,372,648,954</td><td>17</td><td>9</td><td>2332399</td><td>1446</td><td>2740</td><td>4.33</td><td>$B(1,8)$</td></tr>
<tr><td>Voltages</td><td colspan="7">[(1153078,723)|(986975,1027)(291526,221)(893999,607)(2154142,1302)(1154067,25)
(1708205,1085)(1036769,534)(2231379,1199)]</td></tr>
<tr><td>5,768,971,167</td><td>18</td><td>9</td><td>3367759</td><td>1713</td><td>9322</td><td>4.32</td><td>$B(0,9)$</td></tr>
<tr><td>Voltages</td><td colspan="7">[(1621053,981)(872859,802)(2158265,550)(392427,552)(1120958,386)
(2683002,443)(2928215,1511)(3200951,492)(977384,844)]</td></tr>
<tr><td>8,855,580,344</td><td>19</td><td>9</td><td>4481569</td><td>1976</td><td>931</td><td>3.99</td><td>$B(1,9)$</td></tr>
<tr><td>Voltages</td><td colspan="7">[(136746,988)|(1487891,1)(4095718,1761)(791572,659)(4467635,1389)(4286888,1598)
(1440105,422)(3884901,1150)(1891720,801)(2879913,1257)]</td></tr>
<tr><td>12,951,451,931</td><td>20</td><td>9</td><td>12138193</td><td>1067</td><td>8428</td><td>3.61</td><td>$B(0,10)$</td></tr>
<tr><td>Voltages</td><td colspan="7">[(5536234,882)(11617685,773)(2889167,516)(4254683,1043)(7220351,139)
(5561272,727)(11283835,447)(3173765,889)(7907457,1028)(11973428,583)]</td></tr>
</table>

Table 14: Cayley graphs of diameter 9.