RESEARCH STATEMENT

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Contents

1. Degree/diameter problem

With the appearance of the paper [43], the study of Moore graphs started, and more generally, the study of the degree/diameter problem. *Moore graphs* are graphs whose order attains the Moore bound $M(\Delta, k)$, an upper bound for the maximum number of vertices of a graph with given maximum degree Δ and diameter k.

$$M(\Delta, k) = \begin{cases} 1 + \Delta\ \frac{(\Delta-1)^k - 1}{\Delta - 2} & \text{if } \Delta > 2 \\ 2k+1 & \text{if } \Delta = 2 \end{cases} \tag{1}$$

To avoid trivial cases, from now on, we only consider graphs with maximum degree $\Delta \geq 3$ and diameter $k \geq 2$. Moore graph proved to be very rare; they only exist for diameter 2 [5, 14, 43]. All these proofs relied on graph spectra techniques. *Graph spectra* investigates properties and results in graph theory that can be obtained by applying matrix theory to various matrices associated with a graph.

Having settled the existence or otherwise of Moore graphs, with the exception of $\Delta = 57$ and $k = 2$, researchers became interested in a more general setting: the degree/diameter problem.

Degree/Diameter Problem for the Graph Class $\mathcal{C}$ (DDP). Given a graph class $\mathcal{C}$ and bounds Δ and k for the degree and diameter of any graph in $\mathcal{C}$, obtain the maximum number $N(\Delta, k, \mathcal{C})$ of vertices of a graph of $\mathcal{C}$.

Ideally, we would like to have a Moore-like bound for important graph classes. However, apart from bipartite graphs, and some planar and Cayley graphs, no specific upper bound is known, i.e. for another graph class we must content ourselves with the very general Moore bound.

If we are dealing with the class of all connected graphs, we omit $\mathcal{C}$ in $N(\Delta, k, \mathcal{C})$. Let us call (Δ, k)*-graph* a graph of maximum degree Δ and diameter k.

To tackle DDP three approaches have been traditionally followed:

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(1) Classifications of (Δ, k)-graphs whose order misses the Moore bound by a number e. Here the parameter e is called the *defect* and these graphs are called $(\Delta, k, -e)$-*graphs.*
(2) Tight asymptotic bounds for $N(\Delta, k, \mathcal{C})$.
(3) Constructions of large (Δ, k)-graphs of $\mathcal{C}$, which provide lower bounds for $N(\Delta, k, \mathcal{C})$.

My research has approached DDP for several graph classes from all the three above directions.

1.1. **General graphs.** Graphs of defect $e = 1$ do not exist; see [6, 25, 54]. In 1992 Jørgensen [50] proved that, apart from two non-isomorphic $(3, 2, -2)$-graphs and a unique $(3, 3, -2)$-graph, there are no $(3, k, -2)$-graphs for $k \geq 2$.

Together with Mirka Miller, I classified all $(3, k, -4)$-graphs; defect 3, as any odd defect, is not possible for large enough graphs as these must regular. Our result (Theorem 1.1) was obtained through combinatorial analysis of the cycle structure of $(3, k, -4)$-graphs.

Theorem 1.1 ([67])**.** *With the exceptions of diameter 2 and 3, there is no* $(3, k, -4)$*-graph. Furthermore, for diameter 2 there exist two regular and three non-regular* $(3, 2, -4)$*-graphs, while for diameter 3, there is a unique* $(3, 3, -4)$*-graph.*

In a joint work with Mirka Miller and Ramiro Feria-Purón, I proved many non-existence results about $(\Delta, k, -2)$-graphs, the more important ones being the following.

Theorem 1.2 ([30])**.** *There are no* $(\Delta, k, -2)$*-graphs with even maximum degree* $\Delta \geq 4$ *and diameter* $k \geq 4$*.*

Theorem 1.3 ([30])**.** *With the exceptions of diameter 2, there is no* $(4, k, -2)$*-graph. Furthermore, for diameter 2 there is a unique* $(4, 2, -2)$*-graph.*

Similar results for $(\Delta, 2, -2)$-graphs were obtained in [63, 64, 65]. This was a joint work with Mirka Miller and Minh H. Nguyen.

Algebraic approaches for general graphs: Graphs of cyclic defect. I have also approached DDP through graph spectra.

For the adjacency matrix $A(G)$ of a regular graph G of degree d and order n, assume that the polynomial $p(d, m)(A)$ gives the number of paths of length at most m linking any two vertices in the graph. Then, it can be seen that for $(\Delta, k, -e)$-graphs the equation

$$p(d, k)(A) = J_n + B_e$$

holds, where J_n is the matrix of order n with all entries equal to 1.

When $e = 0$ B_e is the zero matrix of order n, that is, the case of Moore graphs. If $e = 1$ B_e is a matrix with each row or column consisting of exactly one 1 and 0 elsewhere. In the case of $e = 2$ B_e is a matrix with each row or column consisting of exactly two 1's and 0 elsewhere, that is, B_e can be considered as the adjacency matrix of a union of vertex-disjoint cycles. Graphs with defect $e = 2$ with the matrix B_e being the adjacency matrix of a cycle of order n are called *graphs with cyclic defect.*

Knowing that the Mobius ladder satisfied the equation $p(d, 2)(A) = J_n + A(C_n)$ Alan Hoffman asked whether or not there was any other graph satisfying the equation. Here $A(C_n)$ denotes the adjacency matrix of the cycle of order n. In 1987 Fajtlowicz [26] proved that the Mobius ladder was the only such graph of diameter 2.

For diameters greater than 2, together with Charles Delorme, I obtained many important non-existence results for graphs of cyclic defect; some of them are summarised below.

Theorem 1.4 ([18])**.** *Let G be a graph of cyclic defect, degree $d \geq 3$, and diameter $k \geq 3$. Then the following results hold.*

(1) *If $k = 3, 4$ there is no such graph for any $d \geq 3$.*

(2) *The number of graphs of odd degree d is asymptotically bound by $O(\frac{64}{3}d^{3/2})$.*
(3) *If $d = 3, 7$ there is no such graph for any $k \geq 3$.*

The proof of Theorem 1.4 relied on algebraic methods, for instance, on the connection between the polynomials $p(d,k)(x)$ and the classical Chebyshev polynomials, on eigenvalues techniques and on elements of algebraic number theory.

The difficulty of using graph spectra techniques in DDP lies in that these techniques often relies on the full specification of the spectrum of the defect matrix B_e, which is not available for $e \geq 2$.

1.2. **Bipartite graphs.** If $\mathcal{B}$ is the class of bipartite graphs, then we have the bipartite Moore bound $M(\Delta, k, \mathcal{B})$, an upper bound on the number of vertices of a bipartite graph of given maximum degree Δ and diameter k.

$$M(\Delta, k, \mathcal{B}) = \begin{cases} 2\frac{(\Delta-1)^k-1}{\Delta-2} & \text{if } \Delta > 2 \\ 2k & \text{if } \Delta = 2 \end{cases}$$

The rarity of bipartite Moore graphs was settled in [27, 75]. Such graphs exist only if the diameter is 2, 3, 4 or 6. After this characterisation no work had been done on classification of families of bipartite graphs whose number of vertices is close to the bipartite Moore bound. My PhD thesis started the study of such families.

It can be seen that for small defect e , say $e \leq 1 + (\Delta-1) + (\Delta-1)^2 + \ldots + (\Delta-1)^{k-2}$, a bipartite $(\Delta, k, -e)$-graph must be regular of degree Δ. Thus, there are no bipartite graphs of defect 1 with $\Delta \geq 3$ and $k \geq 3$, as with any small odd defect.

For bipartite graphs my contributions, in joint work with several authors, are the following.

Theorem 1.5 ([17, 73]). *With the exception of diameter $k = 3$, there are no bipartite $(\Delta, k, -2)$-graphs for $\Delta \geq 3$ and $k \geq 3$.*

The proof of Theorem 1.5 follows a well-known technique in graph spectra [5, 6, 8]. We first prove that the multiplicities of the eigenvalues of a hypothetical graph satisfy certain inequalities, and based on these inequalities, we derive that certain linear combinations over $\mathbb{Z}$ of two eigenvalues must be integer. But, from another point of view, we can prove that those combinations must be in the open interval $(0, 1)$, thus arriving at a contradiction.

In the case of bipartite $(\Delta, 3, -2)$-graphs we proved the uniqueness of the two known bipartite graphs, a $(3, 3, -2)$-graph and a bipartite $(4, 3, -2)$-graph. We also proved several necessary conditions for the existence of bipartite $(\Delta, 3, -2)$-graphs; the most general being that either Δ or $\Delta - 2$ must be a perfect square. These results appeared in [16].

1.3. **Cayley graphs.** DDP has been studied methodically for Cayley graphs of abelian groups [22], and in this case, optimal results have been obtained only for diameters two and three. Recently, DDP has been considered for some metacyclic groups [62].

Important groups to be studied in DDP are p-groups and generalised dihedral groups. The study of generalised dihedral groups was motivated by the paper [22], where DDP for Cayely graphs of abelian groups was considered. The authors of [22] reduced the abelian case to the study of lattice coverings and tilings of Z^{Δ}. They used the fact that the free Abelian group Z^{Δ} on Δ generators is a direct product of the set of integers. Lattice coverings are better known for $\mathbb{R}^{\Delta}$ than for $\mathbb{Z}^{\Delta}$, so this approach mainly relies on approximating real coverings by integer coverings. I feel that lattice coverings and tilings of the Euclidean space, as applied in [22], could be extended to generalised dihedral groups. It is my hope to address the following problem.

Problem 1. *Solve the degree/diameter problem for Cayley graphs of generalised dihedral groups.*

It is known that p-groups play a fundamental role in understanding the structure of important families of groups such as nilpotent groups. I expect that investigations of Problem 1 will provide tools to attack DDP for Cayley graphs of p-groups.

Problem 2. *Solve the degree/diameter problem for Cayley graphs of p-groups.*

My long term aims in this area consist of making substantial contributions to DDP for the class Cay of Cayley graphs, specifically for nilpotent and solvable groups. The emphasis will be on asymptotic bounds; that is, what can we say about $N(\Delta, k, Cay)$ for a fixed value of Δ and large values of k, or vice versa.

1.4. **Minor-closed graph classes.** My research in this area focuses on asymptotic bounds for the order of minor-closed graph classes. In some cases much more precise results are probably possible. In this area I collaborate with Eran Nevo and David Wood.

A well-known result by Jordan [49] implies that every tree of maximum degree Δ and fixed diameter k has at most $(2+o(1))(\Delta-1)^{\lfloor k/2\rfloor}$ vertices. For a graph class $\mathcal{C}$, we define $N(\Delta, k, \mathcal{C})$ to be the maximum order of a graph in $\mathcal{C}$ with maximum degree $\Delta \geq 3$ and diameter $k \geq 2$. We say $\mathcal{C}$ has *small order* if there exists a constant c and a function f such that $N(\Delta, k, \mathcal{C}) \leq c(\Delta-1)^{\lfloor k/2\rfloor}$, for all $\Delta \geq f(k)$. The class of trees is a prototype class of small order.

Little is known about bounds for the sizes of minor-closed graph classes. We first look at the class $\mathcal{P}$ of planar graphs. Fellows, Hell, and Seyffart [28, 29] constructed planar graphs of order approximately $\frac{9}{2}\Delta^{\lfloor k/2\rfloor}$ for odd diameter and approximately $\frac{3}{2}\Delta^{\lfloor k/2\rfloor}$ for even diameter. They also provided an upper bound of $(6k+3)(2\Delta^{\lfloor k/2\rfloor}+1)$ for any diameter $k \geq 2$.

Tishchenko [76] improved the above upper bound to $\frac{3}{2}\Delta^{\lfloor k/2\rfloor}$ whenever $\Delta \geq 6(6k+1)$ and k is even. From Tishchenko's work the next problem naturally follows.

Let $\mathcal{G}_\Sigma$ denote the class of graphs embeddable in a surface[1] Σ of Euler genus g. In the case of diameter 2 Knor and Širáň [53] generalised an earlier result by Hell and Seyffarth [42] which stated that $N(\Delta, 2, \mathcal{P}) = \lfloor\frac{3}{2}\Delta\rfloor + 1$. Knor and Širáň proved the following.

$$N(\Delta, 2, \mathcal{G}_\Sigma) = N(\Delta, 2, \mathcal{P}) = \lfloor\frac{3}{2}\Delta\rfloor + 1. \tag{2}$$

With respect to graphs embeddable in a surface with Euler genus g, Šiagiová and Simanjuntak [78] showed that, for any diameter and absolute constants c_1 and c_2,

$$c_1\Delta^{\lfloor k/2\rfloor} \leq N(\Delta, k, \mathcal{G}_\Sigma) \leq c_2 k g \Delta^{\lfloor k/2\rfloor}.$$

Eran Nevo, David Wood and I proved Theorem 1.6 below, which shows that the class of graphs embedded in a fixed surface Σ has small order.

Theorem 1.6 ([71, Thm. 1])**.** *There exists an absolute constant c such that, for every surface Σ of Euler genus g,*

$$N(\Delta, k, \mathcal{G}_\Sigma) \leq \begin{cases} c(g+1)(\Delta-1)^{\lfloor k/2\rfloor} & \text{if } k \text{ is even and } \Delta \geq c(g^{2/3}+1)k, \\ c(g^{3/2}+1)(\Delta-1)^{\lfloor k/2\rfloor} & \text{if } k \text{ is odd and } \Delta \geq 2k+1. \end{cases}$$

[1] A *surface* is a compact (connected) 2-manifold (without boundary). Every surface is homeomorphic to the sphere with h handles or the sphere with c cross-caps [69, Thm 3.1.3]. The sphere with h handles has *Euler genus* $g := 2h$, while the sphere with c cross-caps has *Euler genus* $g := c$. For a surface Σ and a graph G embedded in Σ, the (topologically) connected components of $\Sigma - G$ are called *faces*. A face homeomorphic to the open unit disc is called *2-cell*, and an embedding with only 2-cell faces is called a *2-cell embedding*. Every face in an embedding is bounded by a closed walk called a *facial walk*.

In [71] we also proved a lower bound on $N(\Delta, k, \mathcal{G}_\Sigma)$ for odd $k \geq 3$ (see [31] for a more complicated construction that gives the same asymptotic lower bound.) Let g be the Euler genus of Σ. It follows from the Map Colour Theorem [69, Thm 4.4.5, Thm. 8.3.1] that K_p embeds in Σ where $p \geq \sqrt{6g+9}$. Let T be the rooted tree such that the root vertex has degree $\Delta - p + 1$, every non-root non-leaf vertex has degree Δ, and the distance between the root and each leaf equals $(k-1)/2$. Observe that T has $(\Delta - p + 1)(\Delta - 1)^{(k-3)/2}$ leaf vertices. For each vertex v of K_p take a copy of T and identify the root of T with v. The obtained graph embeds in Σ, has maximum degree Δ, and has diameter k. The number of vertices is at least $p(\Delta - p + 1)(\Delta - 1)^{(k-3)/2}$. It follows that for odd k, for all $\epsilon > 0$ and sufficiently large $\Delta \geq \Delta(g, \epsilon)$,

$$N(\Delta, k, \mathcal{G}_\Sigma) \geq (1-\epsilon)\sqrt{6g+9}\,(\Delta - 1)^{(k-1)/2}. \tag{3}$$

This lower bound is within a $O(g)$ factor of the upper bound in Theorem 1.6. Moreover, combined with the above upper bound for planar graphs, this result solves an open problem by Miller and Širaň [68, Prob. 13]. They asked whether Knor and Širáň's result could be generalised as follows: is it true that, for each surface Σ and for each diameter $k \geq 2$, there exists $\Delta_0 := \Delta_0(\Sigma, k)$ such that $N(\Delta, k, \mathcal{G}_\Sigma) = N(\Delta, k, \mathcal{P})$ for $\Delta \geq \Delta_0$? We now give a negative answer to this question for odd k. Equation (3) says that $N(\Delta, k, \mathcal{G}_\Sigma)/(\Delta - 1)^{\lfloor k/2 \rfloor} \geq c\sqrt{g} + 1$, while Theorem 1.6 with $g = 0$ says that $N(\Delta, k, \mathcal{P})/(\Delta - 1)^{\lfloor k/2 \rfloor} \leq c'$, for absolute constants c and c'. Thus $N(\Delta, k, \mathcal{G}_\Sigma) > N(\Delta, k, \mathcal{P})$ for odd $k \geq 3$ and g greater than some absolute constant.

Eran Nevo, David Wood and I believe the asymptotic value of $N(\Delta, k, \mathcal{G}_\Sigma)$ is closer to the lower bound in Equation 3 than to the upper bound in Theorem 1.6.

Conjecture 1. *There exist a constant c and a function $\Delta_0 := \Delta_0(g, k)$ such that, for $\Delta \geq \Delta_0$,*

$$N(\Delta, k, \mathcal{G}_\Sigma) \leq \begin{cases} c(\Delta - 1)^{\lfloor k/2 \rfloor} & \text{if } k \text{ is even} \\ c(\sqrt{g}+1)(\Delta - 1)^{\lfloor k/2 \rfloor} & \text{if } k \text{ is odd.} \end{cases}$$

A generalisation to the class $\mathcal{G}_H$ of H-minor-free graphs, with H a fixed graph, was studied in [82]. The current best upper bound of

$$N(\Delta, k, \mathcal{G}_H) \leq 4k(c|H|\sqrt{\log |H|})^k \Delta^{\lfloor k/2 \rfloor}$$

was given in [82, Sec. 4]. Note that if H is planar, then $\mathcal{G}_H$ has bounded treewidth, and thus has small order [82, Thm. 12].

1.5. **Constructions of large graphs.** My first result in this area was the construction of many new largest known graphs of diameter 6 [74], joint work with José Gómez, Mirka Miller and Hebert Pérez-Rosés. The construction was done by *graph compounding*, a technique replacing one or more vertices of a graph with elements of a set of graphs, and then rearranging edges of the resulting graph appropriately to keep the smallest possible maximum degree and diameter. See the table of the largest know graphs in [60].

Methodology in this area includes techniques and concepts, such as graph compounding, vertex duplication, Kronecker product, the voltage assignment technique and polarity graphs; see [61]. Loz and I used [61] these approaches to produce new largest known graphs of maximum degree $17 \leq \Delta \leq 20$ and diameter $2 \leq k \leq 10$; see also [60].

In the construction of large graphs the most successful approach, borrowed from algebraic topology, has been the voltage assignment technique [61, 68]. The *voltage assignment technique* takes a base graph, a group and some generators of the group (*voltage assignments*), and constructs a new, larger graph [37, Ch. 2]. The voltage assignment technique has been used successfully by me and others to generate large graphs with bounded degree and diameter for both general graphs [59, 61, 68] and graphs embedded in surfaces [37, Ch. 4]. The difficulty of using the technique lies in that conditions on the group and the voltage assignments to produce large graphs of a given class are hard to come by. My research attempts to provide methodologies to generate such graphs, especially in the case of Cayley graphs.

In the course of this investigation many subproblems of theoretical importance are likely to appear. For instance, not all graphs can be obtained via the voltage assignment technique. The technique can generate only graphs admitting a non-trivial *semiregular automorphism* (one fixing no vertex of the graph). While all Cayley graphs can be obtained, it is conjectured that so can all vertex-transitive graphs (the *Polycirculant Conjecture* [55]). The Polycirculant Conjecture has been suggested by several authors, including Dragan Marušič and Mikhail Klin.

As far as I am concerned the Polycirculant Conjecture is not settled for *distance-regular graphs*, a combinatorial generalisation of distance-transitive graphs. Any distance-transitive graph is distance-regular, but the converse is not true. In view of the recent settlement of the conjecture for distance-transitive graphs [57], the next natural and important problem is to prove the existence of a semiregular group of automorphisms for vertex-transitive distance-regular graphs. The vertex-transitivity condition cannot be dropped, because there are examples of distance-regular graphs with trivial automorphism groups. I am therefore interested in the following problem.

Problem 3. *Prove the existence of a semiregular group of automorphisms for vertex-transitive distance-regular graphs.*

2. Maximum degree & Diameter Bounded Subgraph Problem

I now consider a straightforward generalisation of the well-known degree-diameter problem (DDP), the Maximum degree & diameter bounded subgraph Problem (MaxDDBS). In this area I collaborate with Hebert Pérez-Rosés.

Max Degree-Diameter Bounded Subgraph Problem for the Graph Class $\mathcal{C}$ (MaxDDBS). Given a network class $\mathcal{C}$, a host graph G of the class $\mathcal{C}$, and numbers d and k, find a largest subgraph of G, in terms of number of nodes, with maximum degree d and diameter k.

In DDP the host graph is simply a sufficiently large complete graph. It is quite surprising that research on MaxDDBS started only very recently in [15].

MaxDDBS has only been studied for the case when the host graph is a common parallel architecture, such as the mesh, the hypercube and the honeycomb grid. The case of the mesh and the hypercube as host graphs were treated in [15].

The paper [66] also considered the mesh as a host graph, refining the upper bounds given in [15] for the order of a largest subgraph for arbitrary $k \geq 1$, and improving the lower bounds on the cases ($k = 3$, $d = 4$) and ($k = 2$, $d = 3$). The honeycomb grid has been studied very recently in [44]. There the authors provided upper and lower bounds for optimal graphs, with special emphasis on the dimensions 2 and 3.

In a nutshell the general problem for MaxDDBS can be stated as follows.

Problem 4 (General problem for MaxDDBS)**.** *Develop original techniques and methodologies for constructing large subgraphs of a given host graph, subject to upper bounds on the degree and the diameter.*

Let us break down this problem into subproblems. Previous works in this area focused on studying the problem for three types of host networks of practical importance: the mesh (or grid), the honeycomb or hexagonal network, and the hypercube. The main results so far included upper bounds for the order of largest subgraphs, and constructions providing lower bounds. This research can be extended naturally to other types of regular networks arising in practice.

Problem 5. *Solve MaxDDBS for a wide range of graph classes.*

We now turn to problems with a computer science flavour. MaxDDBS is NP-hard. Restricting the search to only one constraint (either to the degree or the diameter) already ensures NP-hardness [48, pp. 4-5]. Hence, MaxDDBS is intractable by exhaustive search methods. Moreover, it is not in APX, meaning that it cannot be approximated in polynomial time with a constant approximation ratio. Note that the Maximum Clique Problem, MaxDDBS with diameter $k = 1$, is not even in APX. However, MaxDDBS is amenable to heuristic methods, data reduction techniques, etc. The paper [15] presents a first heuristic approximation algorithm for dealing with MaxDDBS, and analyses its performance theoretically and empirically. There is still plenty of room for improvement here.

Problem 6. *In the context of MaxDDBS study the performance of several heuristic techniques that have been applied successfully to solve other related problems, such as the Degree-Constrained Minimum Spanning Tree [34], and the Maximum Diameter-Bounded Subgraph [3].*

Even though MaxDDBS is intractable and difficult to approximate in general, there must be classes of networks having lower complexity in MaxDDBS. It is important to identify those classes, and devise specific algorithms for them. A first approach to identify some tractable classes consists of splitting the problem into two stages: The Degree-Constrained Subgraph, and the Maximum Diameter-Bounded Subgraph.

Problem 7. *Identify graph classes for which MaxDDBS has lower complexity.*

Some tractable classes have already been identified for those subproblems (e.g. [1, 3]), and are potential candidates for tractable classes in MaxDDBS.

Finally, it is important to determine the position of MaxDDBS in other complexity-class hierarchies, such as the parallel, randomised, and parameterised complexity hierarchies, etc. Our hope is that eventually, this research direction will shed some light on the complexity of the classical Degree-Diameter Problem (DDP), which is unknown to-date. To settle DDP complexity is arguably the most important problem in this area.

Problem 8. *Establish the complexity of DDP.*

3. Hamiltonicity-like properties in graphs

With respect to hamiltonicity-like properties in graphs, we concentrate on vertex-transitive graphs. Vertex-transitive graphs have received much attention due to a Lovász's conjecture on the existence of hamiltonian cycles in all but a few vertex-transitive graphs [56]. With the aim of shedding light on Lovász's conjecture, some relaxations of it have emerged. My research in this area focuses on one such relaxation.

Problem 9 (2WPVT)**.** *Does any connected vertex-transitive graph have a* 2*-walk?*

Problem 9 was proposed by Bojan Mohar (`http://www.fmf.uni-lj.si/~mohar/Problems/P0401VTHamiltonicity.html`).

Bibliography on Lovász's conjecture, 2WPVT and related problems is vast, and the surveys [7, 56] can account for it.

A *k-walk* is a closed walk visiting each vertex of the network at most k times. The concept of k-walk generalises and relaxes the concept of hamiltonian cycle; a hamiltonian cycle is a 1-walk. Settling the existence of 2-walks in vertex-transitive graphs would give further support to Lovász conjecture, while its non-existence would disprove the conjecture, thus the relevance of the study of 2-walks.

What about 3-walks in vertex-transitive graphs? From a result of Win [7, pp. 26] it follows that any vertex-transitive graph admits a *3-tree* (a spanning tree of maximum degree 3). Jackson and Wormald [46] then observed that the existence of a k-tree implies the existence of a k-walk. Thus, the existence of 3-walks in vertex-transitive graphs is settled.

Toughness is a useful necessary condition for the existence of hamiltonian cycles. A graph is *t-tough* if the removal of any cutset S leaves the graph with at most S/t components. The *toughness* of a non-complete graph G is then the maximum t such that G is t-tough. From this definition it follows at once that any t-tough graph is $2t$-vertex-connected. Thus, any connected vertex-transitive graph is at least 1-tough.

There are interesting connections between toughness, hamiltonian cycles and 2-walks. First, there is a conjecture by Chvátal [7, pp. 1] stating that there is a finite t_0 such that all t_0-tough graphs are hamiltonian. Then, a result by Ellingham and Zha [7, Theorem 46] tells us that any 4-tough graph admits a 2-walk.

Results about 2-walks through toughness are not hurdle free. It is known that any bipartite vertex-transitive graph has toughness one, and that, for all degrees ≥ 2, there are non-bipartite Cayley graphs with toughness arbitrarily close to one. See [77] for more details. Nevertheless, it is my opinion that the study of toughness in vertex-transitive graphs, specially in Cayley graphs, can settle the existence of 2-walks in many such graphs.

With regards to 2WPVT, the existence of hamiltonian cycles is known for Cayley graphs of abelian groups, some metacyclic groups and p-groups [56, pp. 5492]. The applicant will focus on the following problems.

Problem 10. *Solve 2WPVT for Cayley graphs of generalised dihedral groups and p-groups.*

Problem 11. *Settle the toughness of Cayley graphs of generalised dihedral groups and p-groups, and connect toughness with the existence of 2-walks in such graphs.*

My long term aims are to settle the existence or otherwise of 2-walks in Cayley graphs for nilpotent and solvable groups, groups with no results on the existence of hamiltonian cycles. In this context I also aim to settle the toughness of such graphs. This research will result in sufficient conditions involving the toughness of a Cayley graph and the existence of 2-walks.

4. Quadratic form representations in Euclidean rings

In 1855 H. J. S. Smith [13] proved Fermat's Two Square Theorem using the notion of palindromic continuants.

Fermat's Two-Square Theorem. *A prime number p is representable by the form x^2+y^2 iff -1 is a quadratic residue modulo p.*

Definition 4.1 (Continuants in arbitrary rings)**.** Let Q be a sequence of elements $(q_1, q_2, \ldots, q_n)$ of a ring R. We associate to Q an element $[Q]$ of R via the following recurrence formula

$$[\,] = 1, [q_1] = q_1, [q_1, q_2] = q_1 q_2 + 1, \text{ and}$$
$$[q_1, q_2, \ldots, q_n] = [q_1, \ldots, q_{n-1}]q_n + [q_1, \ldots, q_{n-2}] \text{ if } n \geq 3.$$

The value $[Q]$ is called the *continuant* of the sequence Q.

Continuants have prominently featured in the literature. For commutative rings many continuant properties are given in [36, Sec. 6.7], while for non-commutative rings a careful study is presented in [79].

The notion of continuant is closely related to the Euclidean algorithm. The Euclidean algorithm outputs a sequence $(q_1, q_2, \ldots, q_n)$ of quotients and a gcd h of m_1 and m_2. A sequence of quotients given by the Euclidean algorithm is called a *continuant representation* of m_1 and m_2 as we have the equalities $m_1 = [q_1, q_2, \ldots, q_n]h$ and $m_2 = [q_2, \ldots, q_n]h$

unless $m_2 = 0$. In other words, $Rm_1 + Rm_2 = Rh$, where Rm denotes the left ideal generated by m.

In [13] Smith deals with the following two questions concerning a prime number p of the form $4k+1$.

> Given a representation x^2+y^2 of p, can we recover a solution z_0 for $z^2+1 \equiv 0 \pmod{p}$?, and vice versa,
>
> Given a solution z_0 for $z^2+1 \equiv 0 \pmod{p}$?, can we recover a representation x^2+y^2 of p?

Smith settled both questions with the use of continuants. Charles Delorme and I considered similar questions in other Euclidean rings, e.g. rings of integers and rings of polynomials over fields of characteristic different from 2.

Problem 12 (From $x^2+gxy+hy^2$ to z^2+gz+h)**.** *If $m = x^2+gxy+hy^2$ and x, y are coprime, can we find z such that z^2+gz+h is a multiple of m using continuants or modifications of continuants?*

Problem 13 (From z^2+gz+h to $x^2+gxy+hy^2$)**.** *If m divides z^2+gz+h, can we find x, y such that $m = x^2+gxy+hy^2$? using continuants or modifications of continuants?*

The question of extending Fermat's Two Squares Theorem to other rings has been extensively considered in the literature; see, for instance, [12, 23, 24, 40, 45, 58, 70, 72]. Perhaps one of the most important extensions of Fermat's Two Squares Theorem was given by Choi, Lam, Reznick and Rosenberg [12], who, in the context of unique factorisation domains, considered representations by forms x^2+hy^2 .

In the past, continuants have been used to find representations of integers by quadratic forms; see, for instance, [11, 41, 80, 81]. In all these works continuants have featured as numerators (and denominators) of continued fractions. For instance, the continuant $[q_1, q_2, q_3]$ equals the numerator of the continued fraction $q_1 + \cfrac{1}{q_2 + \cfrac{1}{q_3}}$, while the continuant $[q_2, q_3]$ equals its denominator.

We emphasise that Smith's approach heavily depends on the existence of a Euclidean-like division algorithm and that if one tries to extend it to other Euclidean rings R, the uniqueness of the continuant representation may be lost. The uniqueness of the continuant representation boils down to the uniqueness of the quotients and the remainders in the division algorithm. This uniqueness is achieved only when R is a field or $R = \mathbb{F}[X]$, polynomial algebra over a field $\mathbb{F}$ [47] (considering the degree as the Euclidean function). Note that in $\mathbb{Z}$ the uniqueness is guaranteed by requiring the remainder to be nonnegative.

In a couple of papers [19, 20] we studied Problems 12 and 13. Some of our results include constructive proofs of Lagrange's Four-Square Theorem, and of the fact that every positive integer has the form $x^2-xy+y^2+z^2-zu+u^2$, $x^2+3y^2+z^2+3u^2$ or $x^2-3y^2+z^2-3u^2$, for integers x, y, z, u. Other results included Proposition 4.1 and constructive proof of it (Remark 4.1).

Proposition 4.1. *Let $R = \mathbb{F}[X]$ be the ring of polynomials over a field $\mathbb{F}$ with characteristic different from 2, and let $-h$ be a (non-null) non-square of $\mathbb{F}$ or a polynomial of degree 1.*

If m divides z^2+ht^2 with z, t coprime, then m is an associate of some x^2+hy^2 with x, y coprime.

Remark 4.1. For the cases covered in Proposition 4.1 we run the Euclidean Algorithm of m by z and stop when a remainder r_{s-1} with degree at most $\deg(m)/2$ is encountered. This will be the $(s-1)$-th remainder, and $(uq_s, u^{-1}q_{s-1}, \ldots, u^{(-1)^{s-2}}q_2)$ are the quotients so far obtained. Then, for a unit u,

$$x = \begin{cases} r_{s-1} & \text{for odd } s \\ u^{-1}r_{s-1} & \text{for even } s \end{cases}$$

$$y = \begin{cases} [uq_s, u^{-1}q_{s-1}, \ldots, u^{(-1)^{s-2}}q_2] & \text{for odd } s \\ u^{-1}[uq_s, u^{-1}q_{s-1}, \ldots, u^{(-1)^{s-2}}q_2] & \text{for even } s \end{cases}$$

As far as we know, Charles Delorme and I were who, for the first time, applied continuants to representations in Euclidean rings other than the integers. The next problem resumes our research line in this area.

Problem 14. *Find new applications of continuants or modifications of them to quadratic form representations in commutative or non-commutative Euclidean rings.*

5. Reconstructing polytopes from their graphs

Research on *convex polytopes* [35, Sec. 16.1], convex hulls of finitely many points in $\mathbb{R}^d$ for some d, is a very active field, partly for the beauty of its mathematics and partly for the number of applications which find polytopes as central objects. Convex polytopes arise in many contexts [35, Chs. 45-47, 50-53, 60-62], for example, in mathematical models of the formation of crystals [35, Ch. 62] and as sets of feasible solutions in linear programming [35, Ch. 45]. The platonic solids are prominent examples of convex polytopes. We only consider convex polytopes, so henceforth we drop the adjective "convex".

My research in this area is concerned with the *combinatorial structure* of polytopes, which is given by partially ordering the polytope faces by the inclusion relation. A *face* of a polytope P is P itself, the *trivial* face, or the intersection of P with a hyperplane which contains P in one of its closed half-spaces, the *proper* faces. The *dimension* of a polytope is the dimension of its affine hull. For a d-dimensional polytope, or d-polytope for short, the 0-faces are called *vertices*, the 1-faces are called *edges*, the $(d-2)$-faces are called *ridges*, and the $(d-1)$-faces are called *facets*. Two polytopes are *equivalent* if there is a bijection between their faces that preserves the inclusion relation. See [83, Ch. 2] or [35, Sec. 16.1.1].

The k-dimensional *skeleton* of a polytope is the set of all its faces of dimension $\leq k$. The 1-skeleton of P is the *graph* $G(P)$ of P, and we think of $G(P)$ as an abstract graph defined on the vertices of P with two vertices being adjacent if they belong to the same edge. Another graph defined on a polytope P is the *dual graph* $G^*(P)$, the graph on the facets of P with two facets being adjacent if they intersect in a ridge. The vertices and edges of $G^*(P)$ are consequently the facets and ridges of P, respectively. This research is about **(1) reconstructing the combinatorial structure of a polytope from its graph or dual graph**, and **(2) if a reconstruction is possible, how efficiently it can be performed**. This interplay between polytopes and their graphs has been investigated in some detail; see [35, Ch. 20] for an overview and [2, 32] for recent examples. For broader applications of graph theory to combinatorial optimisation, see [4].

The research fits within an ambitious research program of **classifying polytope classes according to the minimum $k \geq 1$ such that the polytope can be reconstructed from its k-skeleton**. This classification would give a measure of how combinatorially complex a polytope is. The simplest elements would be polytopes reconstructible from their 1-skeletons (graphs), and the more complex polytopes reconstructible from their $(d-3)$-skeletons [38, p. 229], since every d-polytope can be reconstructed from its $(d-2)$-skeleton [35, Thm. 20.5.21]. Once a reconstruction is settled, algorithmic questions arise on how efficiently it can be done; see [38, Sec.12.4].

In general, the graph or dual graph of a polytope only gives partial information on the structure of the polytope. The graph does not even determine the dimension of the polytope [83, Notes of Ch. 3]. Hence, we always assume the dimension is given. For some classes of polytopes, the graph or dual graph however determines their combinatorial structure; that is, it can be decided if a given induced subgraph of the polytope graph is the graph of a face. This is the case for d-polytopes with $d \leq 3$ and for *simple* d-polytopes, those where every vertex is adjacent to exactly d edges, i.e. the graph is *d-regular*. Blind and Mani [10], and later Kalai [83, Sec. 3.4], showed that a simple polytope is determined by its graph. In one way or another, our research revolves around this result.

One may wonder if the result on simple polytopes is also true for "almost simple" polytopes; a similar question was suggested by Friedman [33, Sec. 8]. Blind et al. [9] thought of an almost simple polytope as a d-polytope having only vertices of degree d or $d+1$. For our purposes, this definition is too weak. Following Grünbaum's comments [38, p. 229], it is possible to construct inequivalent polytopes with the same graph, all having $d+1$ vertices of degree d and d vertices of degree $d+1$. For Friedman [33, Sec. 8], a d-polytope is "mostly simple" if its vertices are mostly of degree d, with only few of degree $d+k$ for some fixed k. The previous discussion suggests calling a d-polytope *almost simple* if, with the exception of at most $d-1$ vertices of degree $d+k$ for some fixed k, all its vertices are of degree d. The first aim explores this generalisation of simple polytopes.

Problem 15. *Investigate whether every almost simple polytope is determined by its dimension and graph. If a reconstruction is possible, provide an efficient algorithm.*

Blind and Mani [10] proved their result for dual graphs of *simplicial* polytopes, duals of simple polytopes [83, Sec. 2.5]. This dual setting makes sense because the graph of a polytope P is isomorphic to dual graph of the dual polytope of P. Simplicial polytopes can also be thought of as those where all proper faces are simplices. *Simplices* [35, Sec. 16.1] are a generalisation of the notion of a triangle or tetrahedron to arbitrary dimensions and the simplest polytopes for a given dimension. From this research, it then becomes natural to investigate whether dual graphs determine the combinatorial structure of *cubical* polytopes, polytopes with all proper faces being *d-cubes* [35, Sec. 16.1], a generalisation of the quadrangle or cube to arbitrary dimensions and the next simplest polytopes. This is precisely a conjecture by Joswig [51, Conj. 3.1]. Support for the conjecture was given in [51] and [35, Thm. 20.5.29]. In terms of reconstructing a cubical polytope from one of its graphs, this conjecture is our only hope, since there are cubical d-polytopes with the same graph as k-cubes for arbitrary $k > d$; see [51, Sec. 3]. For more reconstruction results, refer to [35, Ch. 20].

Problem 16. *Investigate whether every cubical polytope is determined by its dimension and dual graph. If a reconstruction is possible, provide an efficient algorithm.*

Another logical progression of Blind and Mani's work [10] is to assess whether other objects with all proper faces being simplices (e.g simplicial spheres) can be reconstructed from their dual graphs. They in fact suggested this [10, Ques. 1]; see also [35, Prob. 20.5.27]. A *simplicial sphere* is a finite, nonempty collection $\mathcal{C}$ of simplices, with the following properties: $\mathcal{C}$ contains all the faces of the simplices, the intersection of any two simplices is a face of each of them, and $\mathcal{C}$ is homeomorphic to the d-sphere. The elements of $\mathcal{C}$ are its *faces*, and the faces of maximal dimension are its *facets*. The *ridges* of $\mathcal{C}$ are the simplices of dimension one less than a facet. Like polytopes, the ridges and facets of $\mathcal{C}$ give rise to the *dual graph* of $\mathcal{C}$. Note that, unlike polytopes, the *dimension* of $\mathcal{C}$ is defined as the maximum over all dimensions of its facets. Think of a simplicial sphere $\mathcal{C}$ as a "puzzle" where the pieces (the facets of $\mathcal{C}$) are "glued" together along their facets (the ridges of $\mathcal{C}$) so that the final figure is homeomorphic to the d-sphere. Boundary complexes of simplicial 3-polytopes, such as the octahedron and icosahedron, provide readily examples of simplicial 2-spheres. These examples are generalised by *polytopal* simplicial d-spheres, i.e. boundary complexes of simplicial $(d+1)$-polytopes. For more information on complexes and spheres, refer to [83, Chs. 5,8].

Problem 17. *Investigate whether every simplicial sphere is determined by its dimension and dual graph. If a reconstruction is possible, provide an efficient algorithm.*

Aim 17 is true for simplicial 2-spheres, a consequence of Steinitz's Theorem [83, Ch. 4], but it is already open for simplicial 3-spheres. It follows from [10] that this is also true for all polytopal simplicial spheres, since polytopes share the same dual graph with their boundary complexes. This result makes Aim 17 quite natural.

Our last aim explores efficient criteria for telling faces of simple 4-polytopes from subgraphs of the polytope graph. Once we know, by [10], that the facets of simple polytopes can be told from the polytope graph, the quest for an efficient test arises. Kalai's proof [83, Sec. 3.4] gave the first reconstruction algorithm for simple polytopes, albeit exponential in the number of vertices. Later on, Friedman [33], building on a result by Joswig et al. [52, Thm. 1], gave a polynomial-time reconstruction algorithm. We remark that a polynomial-time reconstruction algorithm means that, for a simple polytope, one can decide whether a subset of its vertices is the vertex set of a face in polynomial time. This is the best we can do because a polytope may contain exponentially many faces, and exhibiting them all explicitly may require exponential time.

In the same direction, back in 1970, Perles [39, Sec. 1] conjectured that, for the graph of a simple d-polytope, every induced, connected, $(d-1)$-regular subgraph which is also *nonseparating*, i.e. its removal does not disconnect the polytope graph, is the graph of a facet. A positive answer to Perles's conjecture would immediately give a polynomial-time reconstruction algorithm. Whereas the conjecture holds for a number of classes of d-polytopes [39, Sec. 3], .e.g. for any dimension $d \leq 3$, we now know, thanks to Haase and Ziegler [39, Sec. 4-6], that it is not true in general. Haase and Ziegler [39, Sec. 7] however believe that, for simple 4-polytopes, adding planarity [21, Ch. 4] and a stronger connectivity condition [21, Ch. 3] to the conjecture will make it true.

Problem 18. *Investigate whether, for the graph of a simple 4-polytope, every induced, nonseparating, 3-connected, 3-regular, planar subgraph is the graph of a facet.*

Balinski's Theorem [83, Thm. 3.14] gives a reason for strengthening the connectivity condition. It states that the graph of a d-polytope is d-connected [21, Ch. 3]. Since each facet of a simple d-polytope P is a simple $(d-1)$-polytope, the graph of a facet will therefore be an induced, $(d-1)$-connected, $(d-1)$-regular subgraph of the polytope graph. Adding planarity is justified by Steinitz's Theorem [83, Ch. 4], which states that a graph is the graph of 3-polytope if, and only if, it is 3-connected, 3-regular and planar. In fact, Haase and Ziegler [39, Sec. 7] suggested an even more far-reaching question: is it true that, for the graph of a simple d-polytope, every induced, nonseparating subgraph which is isomorphic to the graph of some simple $(d-1)$-polytope is the graph of a facet? For $d = 4$, Aim 17 is equivalent to this formulation, as every 3-connected, 3-regular, planar graph is the graph of some simple 3-polytope (by Steinitz's Theorem).

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