New Largest Graphs of Diameter 6 ⋆

(Extended Abstract)

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Abstract

In the pursuit of obtaining largest graphs of given degree and diameter, many construction techniques have arisen. Compounding of graphs is one such technique. In this paper, by means of the compounding of complete graphs into the bipartite Moore graph of diameter 6, we obtain two families of $(\Delta, 6)$-graphs. For maximum degree $\Delta > 4$, being $\Delta - 1$ a prime power, the members of these families constitute the largest known graphs of diameter 6.

Keywords: Degree/diameter problem, bipartite Moore graphs, compounding of graphs.

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1 Introduction

The topology of a network is usually modelled by a graph in which vertices represent "nodes" while edges stand for "links" or other types of connections.

When designing networks, several restrictions and/or specifications naturally come out. It seems that the most common limitations lie on the number of connections attached to a node and on the maximum number of links that must be traversed from a node to any other. In graph-theoretical terms, these two parameters stand for the *degree* of a vertex and the *diameter* of the graph respectively.

Let us now recall the well-known Degree/Diameter problem. If links are modelled by undirected edges, the Degree/Diameter problem can be stated in the following way:

> *Degree/Diameter Problem*: Given natural numbers Δ and D, find the largest possible number of vertices $n_{\Delta,D}$ in a graph of maximum degree Δ and diameter at most D.

Research efforts related to the Degree/Diameter problem cover both proofs of nonexistence of graphs of order close to the general upper bounds, known as the Moore bounds and constructions of ever larger graphs. One outcome of such constructions is to improve lower bounds on the maximum possible order of graphs for given D and Δ, $n_{\Delta,D}$.

Presently, compounding of graphs is one of the most important techniques for the construction of large graphs of given degree and diameter. The compounding technique was introduced by Bermond, Delorme and Quisquater [1].

In [7], Quisquater proposed a compounding method which changes a single vertex from a bipartite Moore graph for a suitable complete graph. However,

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in a bipartite Moore graph, it is possible to replace several vertices with copies of an adequate complete graph. This idea is a result of an evolutive work started by Fiol, Serra and Gómez [5] in 1993. Afterwards, using the bipartite Moore graphs of diameter 6, some improvements of this approach have been put forward in [3,6] and recently in [4].

In this paper, the technique of compounding of graphs is used to produce two new families of $(\Delta, 6)$-graphs. We again extend the approach mentioned in the previous paragraph. The members of these families, for $\Delta - 1$ a prime power and $\Delta > 4$, are the new largest $(\Delta, 6)$-graphs. Consequently, their orders improve various entries in the table of the largest (Δ, D)-graphs; see [8].

2 Basic Definitions

Considering that $[S]^k$ denotes the set of all k-subsets of S, a graph is a pair $G = (V, E)$ of sets satisfying $E \subseteq [V]^2$, where $V \cap E = \emptyset$. The elements of V and E are the *vertices* and *edges* of the graph G respectively.

The vertex set of a graph G will be denoted by $V(G)$, and its edge set by $E(G)$. The number of vertices of G represents the order of G, and is denoted by $n(G)$.

In an edge e associated with a pair of vertices u and v, we say that u and v are *neighbors* and *incident* with e; and that e is *incident* with u and v. The set of neighbors of a vertex v in G is denoted by $N(v)$. The *degree* of a vertex v, denoted by $d(v)$, is the number of edges incident with v. The number $\Delta(G)= \max\{d(v)|v \in V\}$ is the *maximum degree* of G.

The minimum length of a cycle in a graph G is the *girth* of G and is denoted by $g(G)$. Let u, v be two vertices in a graph G. The length of the shortest $u - v$ path in G, denoted by $d(u, v)$, is the *distance* in G of u and v. The *diameter* of a graph G, denoted by $D(G)$, is the longest distance between any two vertices in G.

Let $G = (V, E)$ be a graph. G is called *bipartite*, if it is possible to partition V into 2 classes such that every edge has its ends in different classes and the vertices in the same class are not neighbors.

A (Δ, D)*-graph* is any graph of maximum degree Δ and of diameter at most D.

We now define compounding of graphs. Let $S = \{G_1, G_2, \ldots, G_k\}$ be a set of graphs. Each element of S is called a *source graph*. Let $G = (V, E)$ be a graph, called the *base graph*. In addition, let $\hat{G} = (\hat{V}, \hat{E} = \emptyset)$ be a subgraph of G, formed by all those vertices of G to be replaced during the compounding process. We will call $\hat{G}$ the *replaced graph*. Finally, let f be a mapping from $\hat{V}$ to $\{1, 2, \ldots, k\}$.

The compounding of S into G will be denoted by $G(S)$. We define it by means of two steps as follows:

Step 1: Every vertex $v \in \hat{V}$ is replaced by the graph $G_{f(v)} \in S$. The set of added vertices (vertices coming from $G_{f(v)}$) is denoted $\hat{V}(G_{f(v)})$.

Step 2: The edges incident to $v \in \hat{V}$ are distributed among the vertices of $G_{f(v)}$. Note that this step introduces certain degree of ambiguity, so further specifications will often be needed.

3 Bipartite Moore Graphs

The *bipartite Moore bound*, i.e., the maximum number $B_{\Delta,D}$ of vertices in any bipartite graph of maximum degree Δ and diameter at most D, was presented by Biggs [2]:

$$B_{\Delta,D} = \frac{2(\Delta-1)^D - 1}{\Delta - 2} \text{ , if } \Delta > 2.$$

Bipartite graphs attaining this bound are called *bipartite Moore graphs*. Bipartite Moore graphs are Δ-regular graphs and exist only if $D = 2, 3, 4$ or 6. For $D = 2$, they are the complete bipartite graphs of degree Δ ($K_{\Delta,\Delta}$). For $D = 3, 4$ and 6, they have only been constructed if $\Delta - 1$ is a prime power [2]. On the other hand, for $D = 3$, there are values of Δ with no bipartite Moore graphs.

In a bipartite Moore graph G, the following assertions hold:

(i) The girth of G is equal to $2D$.

(ii) Given two vertices x and y, if $d(x, y) = D$, then there exist Δ disjoint paths joining x and y.

4 New Large Compound Graphs of Diameter 6

The process of construction of these graphs is divided into two parts. The first part is a compounding operation and the second one sets up new adjacency rules.

To describe such graphs, we will use the definition of compounding of graphs introduced in Section 2. We base our illustration method on a step-by-step construction that shows a sequence of operations required to define these compound graphs.

Note that the order of any compound graph G using a complete graph K_h as source graph and a subgraph $\hat{G} = (\hat{W}, \hat{E} = \emptyset)$ of G as replaced graph is:

$$n(G) = n(\text{base graph}) + |\hat{W}|(h-1). \tag{1}$$

The compounding part is the same as that described in papers [3,6,4]. We have also kept the same notation and terminology in order to achieve a faster understanding of the idea. However, some conditions to set up the new adjacency rules have changed.

Part 1: Compounding Operation

Source Graphs: Given a positive integer Δ, $G_1 = K_h$, where K_h is the complete graph of order h, for $h \leq \Delta - 1$.

Base Graph: Bipartite Moore graph H_q of diameter 6. A graph $H_q = (V \cup W, E)$ has degree $\Delta = q + 1$, being q a prime power.

Replaced Graph: Let R be the subgraph of $G = H_q$ depicted in Figure 1. Let x be a vertex of V and let $N(x) = \{x_0, x_1, \ldots, x_{\Delta-1}\}$. We now define the following sets:

(i) $W^0 = N(x) - \{x_{\Delta-1}\}$.

(ii) $V^0 = N(W^0) - \{x\} - \displaystyle\bigcup_{i\in\{0,\ldots,\Delta-2\}} \{x_{i\Delta-2}\}$.

(iii) $W' = N(V^0) - W^0 - \displaystyle\bigcup_{\substack{i \in \{0,\ldots,\Delta-2\} \\ j \in \{0,\ldots,\Delta-3\}}} \{x_{ij\Delta-2}\}$.

The set $W' \subset W$ is called the *set of replaceable vertices* and is highlighted in Figure 1.

Given a subset $\hat{W}$ of W', the replaced graph will then be $\hat{G} =(\hat{W}, \hat{E} = \emptyset)$.

Each vertex x_{ijk} of $\hat{W}$ will be replaced by a complete graph K_h, denoted by $K_h^{(ijk)}$, with vertex set $V(K_h^{(ijk)}) = \left\{y_0^{(ijk)}, \ldots, y_{h-1}^{(ijk)}\right\}$.

The set of added vertices is denoted by $\hat{W}(K_h)$. Therefore, $\hat{W}(K_h) = \displaystyle\bigcup_{x_{ijk}\in\hat{W}} V(K_h^{(ijk)})$.

Mapping f:

$$f(x_{ijk}) = 1 \qquad \text{for all } x_{ijk} \in \hat{W}$$

Specifying Step 2: The incident edges to each vertex $x_{ijk} \in \hat{W}$ are joined to the vertices of $K_h^{(ijk)}$, in such a way that each vertex of $K_h^{(ijk)}$ is incident

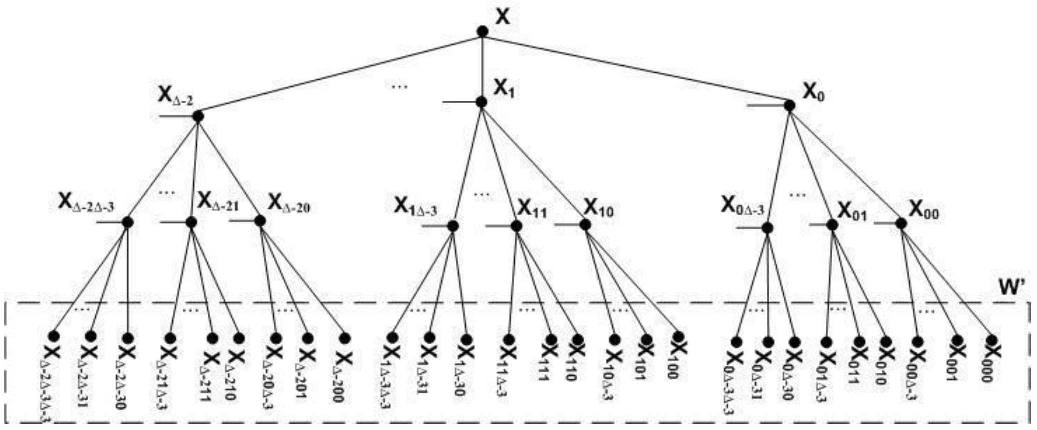

Fig. 1. The subgraph R of H_q to be modified.

at least to one of these edges.

Given a vertex $x_{ijk} \in \hat{W}$ and the complete graph $K_h^{(ijk)}$ replacing it, we call $y_0^{(ijk)}$ the vertex with x_{ij} as neighbor. Then $y_0^{(ijk)}$ is connected at least to one other neighbor of x_{ijk}.

Resulting Graph: $H_q^0(K_h)$.

Part 2: Introduction of New Edges

The set of edges to be added in this part is denoted by $\hat{E}(K_h)$. Therefore, we define the graph $\hat{G}(K_h) = (\hat{W}(K_h), \hat{E}(K_h))$.

We need to introduce new notations and terms. We are now interested in defining the replaced graph $\hat{G} = (\hat{W}, \hat{E})$ completely, that is, to define its edge set. The edge set can be defined as follows: For $x_{ijk}, x_{rst} \in \hat{W}$; $(x_{ijk}, x_{rst}) \in \hat{E} \leftrightarrow \exists\ \alpha, \beta \in \{0, \ldots, h-1\}$ such that $(y_\alpha^{(ijk)}, y_\beta^{(rst)}) \in \hat{E}(K_h)$.

We will call a *block* any set formed by copies of $K_h^{(ijk)}$. However, we will mainly use blocks obtained by copies of $K_h^{(ijk)}$, for a fixed i. That is, $K_h^{(ijk)}$ and $K_h^{(ist)}$ will belong to the same block, for some numbers j, k, s and t not necessarily distinct.

Now we will describe the proposal to add the new edges in our approach.

Origin Graph: $H_q^0(K_h)$.

Condition 1: There should be one edge between each pair of copies $K_h^{(ijk)}$ and $K_h^{(ijt)}$, for $k \neq t$.

Condition 2: The cardinal of each block C_b is the same, that is, each block contains the same number of copies of $K_h^{(ijk)}$.

Condition 3: The distance between any two copies of $K_h^{(ijk)}$ inside a block should be at most 3. Particularly, the distance between any two copies $K_h^{(ijk)}$ and $K_h^{(ist)}$ should be at most 3.

Condition 4: Given the number of blocks N_B, with $N_B \leq \Delta - 1$, each graph $K_h^{(ijk)}$ should have at least $N_B - 1$ free edges to be used for connections inter-blocks.

Condition 5: The distance between any two copies $K_h^{(ijk)}$ and $K_h^{(rst)}$ should be at most 4, for $i \neq r$. Equivalently, the distance between any two copies of $K_h^{(ijk)}$ from different blocks should be at most 4.

Condition 5a: The distance between any vertex y_α^{ijk} and any copy of type K_h^{rst}, $i \neq r$, should be at most 5.

Resulting Graph: $H_q(K_h)$.

Remark 4.1 Conditions 5 and 5a can indistinctly be used.

Remark 4.2 Following Equation (1), we obtain that

$$n(H_q(K_h)) = n(H_q) + N_B C_b (h-1). \tag{2}$$

Lemma 4.3 *Graph $\hat{G}(K_h) = (\hat{W}(K_h), \hat{E}(K_h))$ has diameter 6.*

Proof (Sketch) The proof can be derived from the previous conditions. □

Theorem 4.4 *Graphs $H_q(K_h)$ have diameter 6.*

Due to space limitations, the proof of this theorem is not included in this Extended Abstract.

As a consequence of Theorem 4.4, two new largest graphs arise, $H_4(K_3)$ and $H_5(K_4)$. Following Equation (2), we obtain that the order of the new graph $H_4(K_3)$ is $n(H_4) + 3 \cdot 7 \cdot 2$, that is, 2772. An analogous reasoning shows that the order of the new graph $H_5(K_4)$ is $n(H_5) + 5 \cdot 7 \cdot 3$, that is, 7917.

4.1 Constructing $H_q(K_h)$ Graphs of Larger Maximum Degrees: A Different Approach

By modifying the constructive process of a family of graphs suggested by Gómez [4], we obtain new graphs of diameter 6 and maximum degree Δ, for $\Delta - 1$ a prime power and $\Delta > 6$. The orders of these graphs are the largest known for their respective maximum degrees.

5 Conclusion

In this paper, we have presented two new families of compound graphs of maximum degree Δ and diameter 6, called $H_q(K_h)$. Members of these families, for $\Delta - 1$ a prime power and $\Delta > 4$, constitute the largest $(\Delta, 6)$ graphs. Table 1 shows the orders of the new graphs for $\Delta \leq 14$.

Note that the previous graphs were obtained in [6] and in [4], and we keep the same terminology as these papers.

Maximum Degree (Δ)	$H_q(K_h)$	*Previous Order*	*New Order*
5	$H_4(K_3)$	2766	**2772**
6	$H_5(K_4)$	7908	**7917**
9	$H_8(K_6)$	75828	**75893**
12	$H_{11}(K_8)$	359646	**359772**
14	$H_{13}(K_{11})$	816186	**816294**

Table 1
New orders of the largest graphs $H_q(K_h)$ for $\Delta \leq 14$.

References

[1] Bermond,J.-C., C. Delorme and J.-J. Quisquater, *Grands graphes non dirigés de degré et diamètre fixés*, Ann. Discrete Math. **17** (1982) 65–73.

[2] Biggs, N.I., "Algebraic Graph Theory," Cambridge University Press, Second Edition, Great Britain, 1993.

[3] Comellas, F. and J. Gómez, *New large graphs with given degree and diameter*, Graph theory, Combinatorics and Applications: Proceedings of the seventh quadrennial international conference on the theory and applications of graphs, vol. 1, 1995, pp. 221–233.

[4] Gómez, J. *On large* $(\Delta, 6)$*-graphs*, Networks **46** (2005), 82-87.

[5] Gómez, J., M.A. Fiol and O. Serra, *On large* (Δ, D)*-graphs*, Discrete Mathematics **114** (1993) 219–235.

[6] Gómez, J., I. Pelayo and C. Balbuena, *New large graphs with given degree and diameter six*, Networks **34** (1999) No. 2, 154–161.

[7] Quisquater, J.J., "Structures dinterconnexion: Constructions et applications," Thse d'etat, LRI, Universit de Paris Sud, Orsay, July 1987.

[8] "The (Δ, D)-problem for graphs," http://www-mat.upc.es./grup_de_grafs/table_g.html.